Reasoning about assertions, obligations and causality: on a categorical semantics for a logic for pragmatics

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Declaration

The work presented here is, apart from explicitly cited material, my own.

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Abstract

The aim of the logic for pragmatics considered in this work is to provide a logical framework that formalises reasoning about the pragmatic forces with which a sentence may be uttered. The concept of pragmatic or illocutionary force comes from speech act theory and plays a crucial role also in certain branches of artificial intelligence, in particular in the development of communication protocols for software agents. Instead of considering the full-blown theory of speech acts, we focus on speech acts that either have the pragmatic force of an assertion or the pragmatic force of an obligation, and on how these speech acts may be related to each other. In particular, we are interested in a principle proposed by Bellin and Dalla Pozza that allows one to promote acts of obligations through causal chains of acts of assertions. The main achievement of this thesis is a sound and complete categorical semantics for a logic for pragmatics incorporating the aforementioned principle. One of the benefits of the proposed semantics is that it allows one to deal with conditional obligations as well, thus extending the framework in a very interesting way. Although the logical framework considered in this work incorporates only two types of speech acts, we hope to be able to show that we have a well-behaved core fragment that can serve as a fruitful basis for further investigations.

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Ma in realtà niente altera niente, e ciò che diciamo o facciamo

sfiora soltanto la cima dei monti nelle cui valli dormono le cose.

(Fernando Pessoa, Il libro dell'inquietudine di Bernardo Soares)

Chapter 1

Introduction

The aim of this thesis is to investigate the proof theory and semantics of various fragments of the logic for pragmatics proposed by Bellin and Dalla Pozza in [BD02]. We are dealing with a system that provides a so-called prescriptive account of deontic logic, i.e. one that takes as its basic assumption that norms or obligations are not simply used to state what ought to be the case but to bring about a certain state of affairs. In other words, the operator used to express the deontic modality cannot be treated in terms of a truth-functional semantics. In the remainder of this introductory chapter we shall give an account of the basic concepts of the logical framework, illustrated by means of an example, discuss some issues concerning the technical contributions of the present work, and provide a brief outline of the thesis.

1.1 Sketch of basic intuitions

One of the key ideas of the framework we are about to investigate is to employ a well known distinction endorsed by the theory of speech acts, namely the one between the pragmatic force of an utterance and its propositional content.

Car chasing scenario To get an idea of what is going on let us introduce a simple example that allows us to illustrate the basic issues in a concrete case. Suppose that a police man enters a taxi and gives to the driver the following order: "Keep close to that red car!". Within this context one may deduce that if the driver of the red car speeds then the taxi driver must speed as well, i.e. one derives the obligation for the taxi driver to speed from the obligation of keeping close to the red car, conditionally upon the fact that the red car speeds. As we shall argue below the formalisation of such a pattern of commonsense reasoning can naturally be done in the logical framework introduced by Bellin and Dalla Pozza in [BD02].

Obligations as prescriptions First of all, let us consider the utterance of the police officer. One should note that it is not a sentence in declarative mood but an imperative. As such it is impossible to assign a truth value to it, since the question of whether "Keep close to the red car!" is true or not simply does not apply. A way out is offered by speech act theory: one can decompose a speech act such as the above imperative into two separate parts, its force and its content. The content consists of a proposition p, i.e. in the example case we would have that p = *taxi keeps close to red car*, whereas the force tells us how the utterance has been used, for instance as the prescription to bring about the state of affairs described by p. Although one can assign truth values to propositions, it is not possible to do so for speech acts since truth values cannot be assigned to actions in general. Nevertheless, one can argue that speech acts are justified or not. If in the above example someone other than a police officer would have given the order then the taxi driver might take the derived obligation to speed as unjustified (and therefore not obey it) since an ordinary costumer does not have the authority to give such orders to him. Hence, considering justification values instead of truth values provides a reasonable starting point for the development of a prescriptive account of deontic logic.

Declaratives as assertions It is worth mentioning that with respect to [BD02] we take it for granted that factual statements may be present in the derivation of an obligation from other obligations, thus making it possible to consider also conditional obligations, i.e. obligations that are justified only if certain circumstances obtain. For instance, the obligation for the taxi driver to speed is justified conditionally upon the fact that the red car speeds. The general problem with factual statements is that we can assign truth values to them, thus threatening the assumption that the logical framework is based on justification values. Again, speech act theory provides a solution: declaratives are simply seen as speech acts that have the force of an assertion. To sum up, the key idea of the logic considered in this thesis is to have operators that allow one to express the pragmatic force of a proposition, thus providing a logical framework for dealing with acts of assertion and obligation. In other words, if propositional logic is conceived as the investigation of the abstract properties of natural language particles such as *not*, *and*, *or*, and *entails* then Bellin and Dalla Pozza's logic for pragmatics may be conceived as the investigation of the ways in which certain utterances are used to do things.

A logic for pragmatics So far we have only seen the basic ingredients of the logical framework. To get a glimpse of the general picture it is worth to be more precise about the nature of justification. As we have seen above justifications may depend heavily on context, e.g. an obligation may be justified or unjustified depending on the authority of the speaker. To get started it seems reasonable to consider only abstract speech acts, i.e. speech acts in which one abstracts away from the particular speech situation, thus leaving aside many of the aspects that play a major role in speech act theory and pragmatics. Although this limits the range of application it is still the case that the logical framework provides a good prescriptive account of deontic logic. With respect to assertions and obligations one has that

1. the act of asserting the proposition p is justified if there is a formal proof or, outside mathematical contexts, some other kind of evidence showing that p is true, and unjustified otherwise;

2. the act of setting the obligation to bring about the state of affairs described by p is justified if there is a proof showing that p holds in every rational normative system, and unjustified otherwise.

Since both the justification of an assertion and an obligation depend crucially on the concept of proof one can simply exploit the well known Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic and therefore claim that the logical framework is essentially governed by the standard laws of propositional intuitionistic logic.

Formalising the example We are now in the position to introduce an inference rule that allows one to derive the conclusion of the above example in a purely formal way. Let us suppose that the following list of propositions is provided: K = *taxi keeps close to red car*, S_r = *red car speeds*, S_t = *taxi speeds*. Then, given that K *and* S_r *entail* S_t (formally written as $K, S_r \Rightarrow_c S_t$) we can apply the inference rule

$$\frac{K, S_r \Rightarrow_c S_t}{(K)^\circ, S_r \Rightarrow_d (S_t)^\circ}$$

to get that $(K)^\circ$ *and* S_r *entail* $(S_t)^\circ$ (formally written as $(K)^\circ, S_r \Rightarrow_d (S_t)^\circ$) where the operator $(-)^\circ$ has the following intended reading: $(p)^\circ$ stands for the obligation to bring about the state of affairs described by p. Note that there are three important differences with respect to the usual modal rule of system **KD**, the standard modal system for deontic logic: (a) we are working in an intuitionistic rather than a classical setting; (b) we are not forced to apply the modal operator $(-)^\circ$ to all the formulae on the lefthand side of $\Rightarrow_d$, thus providing means to express the parameterisation of obligations by the deductive context of the modal rule; (c) we switch from the entailment relation $\Rightarrow_c$ in the premise of the inference rule to the entailment relation $\Rightarrow_d$ in its conclusion, thus getting a flat modality. Having a flat modality means that we cannot apply the modal operator $(-)^\circ$ more than one time. This restriction takes care of the requirement that prescriptive deontic modalities cannot be iterated.

Causality and conditionals There are two further issues that need to be considered. First, in the formalisation of the example we have implicitly assumed that the entailment $K, S_r \Rightarrow_c S_t$ can be treated as an axiom of the logical system. However, in [BD02], Bellin and Dalla Pozza put emphasis on the requirement that an obligation can be derived from other obligations only if in the premise of the inference rule the factual statements on the lefthand side of $\Rightarrow_c$ are causally related to the the factual statement on its righthand side. In other words, one has to replace the entailment $K, S_r \Rightarrow_c S_t$ by $K, S_r, (K \circ S_r) \Supset S_t \Rightarrow_c S_t$, thus making explicit the causal link between the factual statements. Note that, as an immediate consequence, the premise of the extended inference rule

$$\frac{K, S_r, (K \circ S_r) \Supset S_t \Rightarrow_c S_t}{(K)^\circ, S_r, (K \circ S_r) \Supset S_t \Rightarrow_d (S_t)^\circ}$$

can be derived by means of the logical rules for the connectives $\circ$ (relevant conjunction) and $\Supset$ (causal implication). Second, as pointed out earlier it is very natural to think about the conclusion of the inference in terms of conditionals. Hence, instead of letting the parametrisation of obligations be expressed only by the deductive context of the modal rule, we can introduce a new syntax for conditional obligations. From the conclusion of the above instance of the inference rule we can thus derive

$$\frac{(K)^\circ, S_r, (K \circ S_r) \Supset S_t \Rightarrow_d (S_t)^\circ}{(K)^\circ, (K \circ S_r) \Supset S_t \Rightarrow_d (S_t)^\circ /\!/ S_r}$$

where $(S_t)^\circ /\!/ S_r$ is justified only if the obligation $(S_t)^\circ$ is justified conditionally upon the fact that the assertion S_r is justified. Note that by enhancing the logical framework with a new connective $/\!/$ for conditional implication we approach the investigation of an extension of the framework suggested by Dalla Pozza in the conclusion of [Dal97].

Further issues One should note that in this brief exposition we have only scratched the surface of a rather complex philosophical topic. A more in depth account of these issues that provides a concise overview of the historical development of the project of a logic for pragmatics as well as pointers to the relevant literature can be found in chapter 2.

1.2 Some technical background

The main theme of the present work is the investigation of the modal rule introduced above from a proof-theoretical, type-theoretical, and semantic point of view. In this section, we shall first explain what is actually meant by this statement and then move on to dicuss the problems encountered along the way and motivate their solutions.

Semantic aspects For a start it is worth reminding the reader of the work in [BR03]: it provides a Kripke-style semantics for the logic for pragmatics of Bellin and Dalla Pozza that is based on a translation of the logic into bimodal logic. Although this suffices to get a completeness result, it is more desirable to have a semantic analysis that does not depend on such a translation. In particular, getting a completeness result with respect to a categorical semantics provides means to regard other semantics as particular instances thereof. Indeed, later on we shall see a class of Kripke models that induce a particular class of categorical models, the so-called algebraic models. We might thus say that one of the motivations for the present work is the investigation of a semantics for the logic for pragmatics proposed by Bellin and Dalla Pozza that does avoid the shortcomings of the semantics considered in [BR03].

Syntactic aspects We have hinted at the fact that category theory provides us with a powerful tool that allows one to regard various models of a logic as particular instances of the general categorical semantics. The crucial issue here is that one has a close relationship between three different ways of looking at a logical framework from a proof-theoretical perspective, one of these being provided by category theory. The other ways are provided by logic and λ-calculus. This is commonly called the extended Curry-Howard correspondence and the basic relations between the key concepts of each of the formalisms are summarised by the rows of the following table:

logic	λ-calculus	category theory
formulae	types	objects
derivations	terms	morphisms
cut elimination	reductions	equality of morphisms

The Curry-Howard correspondence is so fundamental in theoretical computer science because it provides means to reason about functional programming language, i.e. suitable extensions of the λ-calculus, in terms of logic, thus making it possible to replace informal reasoning by formal arguments. Note, however, that we shall use the above table only as a guideline and rather look for a type-theoretical presentation of the logic for pragmatics: it is characterised by the fact that the distinctions between the various columns of the above table are somehow blurred. More precisely, a type theory consists of a natural deduction system where formulae or types are denoted by terms together with a collection of equations between terms that are induced by certain proof transformations of the natural deduction system.

Layered approach Since [BD02] considers only a sequent calculus system for the logic for pragmatics, a first obstacle that has to be removed is the lack of a natural deduction system. A key step towards the development of such a system is provided in [BR03] where one can find the observation that the derivations of the sequent calculus system given by Bellin and Dalla Pozza can all be brought in some sort of canonical form. We shall extend this idea and replace the double-context approach of [BD02] by an approach that is based on multiple entailment relations. Indeed, the design of the logic as a layered system mimics the definition of pragmatic language provided in the next chapter. As as consequence we have that one can almost immediately derive the natural deduction rules from the sequent calculus rules, thus making it possible to get a type-theoretical presentation of the logic for pragmatics. It is worth pointing out that our approach is inspired by the work on linear and non-linear logic (see for instance [Ben94] or [MMdPR05]) and the logic of linear functors (see for instance [BCS02]).

On the modal rule As mentioned above, the techniques used to investigate and study natural deduction systems for constructive modal logics (see for instance [Bel85] and [BdPR01]) can be adapted in a straightforward manner to the case of the modal rule considered in the previous section. However, whereas one can adapt the standard categorical machinery for most of the rules, this is not anymore the case for the modal rule. The usual categorical approach to modalities (at least if one is interested in modelling proofs rather than provability) consists in providing an endofunctor with certain extra structure such as, for instance, strength. But since we have a flat modality that cannot be iterated an arbitrary number of times this approach is doomed to fail. Although flat modalities arise if one breaks the monoidal adjunction in categorical models for linear and non-linear logic (see for instance [Epp03]), we shall rather follow the lead of [MdPR00] and provide an approach that is based on split $\mathbb{B}$-indexed categories, i.e. functors from $\mathbb{B}^{op}$ to $\mathbf{Cat}$, the category of small categories. The key observation is that the so-called parameters of the modal rule remain unaltered, thus suggesting to use them as the objects of the base category $\mathbb{B}$. As we shall see later on the two morphisms representing the derivation of the premise and the derivation of the conclusion of the rule can be obtained as the images of two functors from $\mathbb{B}^{op}$ to $\mathbf{SCat}$, the category of small semi-categories or categories without identity morphisms. It is worth pointing out that every category can be regarded as a semi-category. Thus, we simply use a slightly more general variant of the standard definition of split indexed category. As a consequence the modal rule can be interpreted as a natural transformation between these two functors, i.e. as a morphism of generalised split indexed categories.

Outline of the thesis As remarked at the end of the previous section, in chapter 2 we provide a gentle introduction to the basic philosophical ideas underlying this thesis. The main goals of chapter 2 are to elaborate in more detail the philosophical issues sketched above and to give a formal definition of pragmatic language. Note that we distinguish three fragments of the pragmatic language: the basic causal-deontic fragment, the extended causal-deontic fragment, and the full intuitionistic fragment. This allows us to present the technical work in a more modular fashion. Most of the remaining chapters are concerned with the basic causal-deontic fragment, i.e. the smallest fragment incorporating the modal rule, its investigation being carried out in chapters 3, 4, 5, and 6. Chapters 3 and 4 serve the purpose to provide some intuitions about the basic causal-deontic fragment of the logic for pragmatics, both from a proof-theoretical and a semantic point of view. In chapter 3 we introduce a sequent calculus system for the logic and show that it satisfies the cut elimination property. In chapter 4 we introduce a Kripke semantics and show that it is sound and complete with respect to the sequent calculus system. Chapters 5 and 6 are the most fundamental ones of the thesis. In chapter 5 we introduce a type theory for the basic causal-deontic fragment of the logic for pragmatics that is derived from the sequent calculus presentation of the system. In chapter 6 we introduce a sound and complete class of categorical models and investigate how the Kripke semantics fits into the abstract framework. Having thus completed the investigation of the basic causal-deontic fragment, we set out in chapters 7 and 8 to explain how similar results can be obtained both for the extended causal-deontic and the full intuitionistic fragment of the logic for pragmatics. In chapter 7 we deal with the extended causal-deontic system and show how the definitions of the previous chapters have to be adapted in order to support the concept of conditional obligation or hypothetical norm in the logical framework. In chapter 8 we deal with the full intuitionistic system and show how the original system of [BD02] is related to the system considered in this thesis. To sum up, one might say that we provide a new presentation of the original system with a well-behaved semantics that can naturally be enhanced with a concept of conditional obligation. We conclude the thesis with some reflections about possible directions for future research.

Chapter 2

Philosophical aspects

The project of a logic for pragmatics is based on a wealth of philosophical ideas. To make these ideas more accessible we briefly sketch the historical development of the project with the aim to introduce the important concepts step-by-step, thus providing a concise overview of the general framework. In short, the main feature of the logic for pragmatics are operators that allow one to express explicitly the force with which a sentence is uttered. In particular, we are interested in sentences or utterances that have the force of an assertion or obligation, consider issues that arise when one tries to relate impersonal acts of assertion and impersonal acts of obligation via a notion of causal implication, and show that the framework can naturally be enhanced with a notion of conditional obligation.

2.1 Towards a logic for pragmatics

Based on ideas from Frege's philosophy of language and ordinary language philosophy, in particular speech act theory, the project of a logic for pragmatics, originated by the philosopher Carlo Dalla Pozza, proposes a formal apparatus that allows one to integrate various logics within the same conceptual framework.

Propositions vs. judgements A cornerstone in the development of philosophy of language is Frege's investigation of aspects related to the concept of thought or, to put it in the terminology of modern logic, proposition.

> Wir unterscheiden [...]:

> 1. das Fassen des Gedankens – das Denken,
> 2. die Anerkennung der Wahrheit eines Gedankens – das Urteilen,
> 3. die Kundgebung dieses Urteils – das Behaupten.[1] [Fre19, p. 62]

For Frege the apprehension of a thought or proposition α roughly corresponds to asking the question of whether it is the case that α or, equivalently, that α is true. In general, one does not know the answer to such a question a priori; rather one has to go through some sort of process that provides means to judge whether α is true. Indeed, in scientific or mathematical discourse such means consist of empirical or formal proofs, respectively. Therefore, only if one has some sort of evidence, perhaps in form of an empirical or mathematical proof, in support of the proposition α, one can assert its truth. In his early work Frege even introduces special signs to distinguish between these aspects in his graphical language (see for instance the discussion in [SL95, pp. 29-35]): he uses $-\alpha$ to denote the content of an assertion, and $\vdash \alpha$ to denote the assertion itself. The conceptual shift from propositions to judgements is thus symbolised by putting a vertical stroke in front of the content sign.

Speech act theory Analogous ideas can be found in the theory of speech acts, a branch of pragmatics heavily influenced by Austin's ordinary language philosophy. Austin's basic insight is that there are cases in which "to say something is" not "always and simply to state something" but rather "to do something" [Aus76, p. 12]. Indeed, take for example the paradigmatic case where a judge declares 'I sentence you to ten years of prison' and the effects caused by this declaration. In particular, Austin considers "the senses in which to say something is to do something" [Aus76, p. 121] and distinguishes

> the locutionary act [...] which has a *meaning*; the illocutionary act which has a certain *force* in saying something; the perlocutionary act which is *the achieving of* certain *effects* by saying something. [Aus76, p. 121]

It is intended that to perform a locution or locutionary act is "roughly equivalent to uttering a certain sentence with a certain sense and reference, which [...] is roughly equivalent to 'meaning' in the traditional sense" [Aus76, p. 109]. Within this context speech act theory is mainly

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[1] This may be translated as follows.

> [...] we may distinguish:
> (1) the apprehension of a thought – thinking,
> (2) the recognition of the truth of a thought – judgement,
> (3) the manifestation of this judgement – assertion. [Fre56, p. 294]

concerned with the study of illocutions or illocutionary acts, broadly understood as "the making of a statement, offer, promise, etc. in uttering a sentence, by virtue of the conventional *force* associated with it" [Lev83, p. 236]. The main feature of the concept of illocution is that one can make explicit the force with which a sentence is uttered, thus providing means to detach the force of an utterance from its content.

> [...] the distinction between illocutionary force and propositional content can in fact be found in [...] Frege. Frege himself placed considerable emphasis on the distinction between the "thought" or proposition, and its assertion or "judgement" as true. [Lev83, p. 242]

As in Frege's graphical language for assertions a speech act can therefore be regarded as a proposition α endowed with a pragmatic operator (such as for example $\vdash$) indicating the force or pragmatic mode of the proposition.

Justification values Another crucial aspect of Austin's work are the so-called felicity conditions: they provide means to determine whether an illocution or speech act is performed successfully. There are various kinds of conditions to consider and [Aus76] provides a full account (see also [Lev83, section 5.1]). Instead of going into the details let us just point out that, roughly speaking, they serve the purpose to guarantee that the speech act under consideration takes place in favourable circumstances and that it follows some specific procedure, usually established by convention. However, if one takes seriously the idea to develop a logic for pragmatics conceived as a logic of speech acts then one has to be clear about how the felicity conditions are tied to the logical framework. Dalla Pozza's logic for pragmatics is based on two assumptions that can be traced back to the work of Frege. First, only impersonal speech acts, i.e. speech acts in which one abstracts away from the particular speech situation, are considered. As a consequence the pragmatic operator $\vdash$ seen above is interpreted as either 'the assertion that —' or 'asserting that —' but not as 'some speaker asserts that —.' That it is reasonable to start from such an assumption also follows from the type/token distinction: for, impersonal speech acts should be regarded as types, and ordinary speech acts as tokens.[2] Second, speech acts are actions and as such they can be either justified or unjustified, i.e. they can be endowed with a justification value. The relationship between felicity conditions and justification values may then be expressed as follows: impersonal speech acts are felicitous or performed successfully only if they are justified. There-

[2]The idea that one can appeal to the type/token distinction was originally proposed by Graham White.

fore, to be justified is a necessary condition for a speech act, may it be impersonal or not, to be felicitous. To establish under which conditions certain kinds of speech acts are justified is one of the main tasks of the logic.

A logic of judgements Starting from the above assumptions and the commonly accepted fact that classical logic is the logic of propositions, Dalla Pozza sets out in [Dal91, DG95] to develop his basic account of a logic for pragmatics. It is best to recall the distinction between propositions and judgements discussed above. A proposition or formula in classical logic α is defined by the following grammar:

$$\alpha := p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha$$

As usual, p ranges over the set of atomic propositions and propositions are interpreted according to their standard truth-functional semantics. A judgement or assertion is the recognition of the truth of a proposition and as such it cannot be endowed with a truth value. But, as we have seen above, we can assign a justification value to it and hence the question of whether there is a logic of judgements in which justification values play the same role as truth values in classical logic arises. Dalla Pozza thesis is that there is such a logic and that it is governed by the laws of intuitionistic logic. For, an impersonal act of assertion $\vdash \alpha$ is justified if and only if there exists a proof (which depending on context may either be empirical or mathematical) of the proposition α. Therefore, the connectives of the logic of judgements can be interpreted in terms of proofs: for instance, that $\vdash \alpha_1$ entails $\vdash \alpha_2$ is justified if and only if there exists a method that transforms a justification of $\vdash \alpha_1$, i.e. a proof of α_1, into a justification of $\vdash \alpha_2$, i.e. a proof of α_2; and similarly for the other connectives. In other words, starting from Frege's insight that judgements and assertions belong to the same subject matter one can exploit the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic to provide an informal semantics for the logic of judgements. To sum up, the logic of judgements can be regarded as an extension of the logic of propositions defined by the following grammar:

$$\eta := \vdash \alpha \qquad \delta := \eta \mid \mathbb{J} \mid \mathbb{U} \mid \delta \cap \delta \mid \delta \cup \delta \mid \delta \supset \delta$$

In Dalla Pozza's terminology a formula δ is called sentential.[3] Sentential formulae are obtained from the class of so-called elementary (sentential) formulae η and the logical constants $\mathbb{J}$ and

[3]Dalla Pozza actually uses the term *assertive formula* in [DG95]. However, when he considers a second pragmatic operator or sign of pragmatic mode for deontic modality in [Dal97] he introduces the term *sentential formula* instead and distinguishes between assertive and normative sentential formulae.

$\mathbb{U}$ by applying the pragmatic connectives $\Cap$, $\Cup$, and $\Supset$ standing for intuitionistic conjunction, disjunction, and implication, respectively. $\mathbb{J}$, the act of assertion that is always justified, and $\mathbb{U}$, the act of assertion that is never justified, are the units for $\Cap$ and $\Cup$, respectively. In the basic account of the logic the class of elementary formulae consists only of impersonal acts of assertion; we shall see below that it can be extended with other kinds of speech acts. An elementary formula $\vdash \alpha$ is characterised by the property that it detaches the operator $\vdash$ indicating the pragmatic mode from the proposition or, to put in Dalla Pozza's terminology, radical part α. Elementary formulae bridge the gap between the logic of propositions, representing essentially semantic aspects of language, and the logic of judgements, representing essentially pragmatic aspects of language.

On the unity of logic Dalla Pozza's basic account of a logic for pragmatics enables one to draw some important philosophical conclusions. First, since sentential formulae have a justification value one can define a concept of pragmatic validity for the logic of judgements that naturally extends the concept of validity for classical logic (see for instance [DG95, section 3]). Second, both classical and intuitionistic propositional logic can be considered as subsystems of the logic. This is achieved by restricting the class of sentential formulae to the class of elementary formulae η for classical logic and to the class of sentential formulae obtained from elementary formulae of the form $\vdash p$ only for intuitionistic logic. The logic can thus be regarded as a unifying framework where both classical and intuitionistic logic can coexist, the main reason being that classical logic, seen as the logic of propositions, and intuitionistic logic, seen as the logic of judgements, have a different subject matter. Third, it is possible to provide a link between the pragmatic and the semantic level of the logic via the Gödel-McKinsey-Tarski interpretation of intuitionistic logic in the classical modal logic S4. To do so one extends the class of radical formulae α with an operator $\Box$ obeying the laws of S4:

$$\alpha := p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \Box\,\alpha$$

Note that the intended reading of the classical modal operator $\Box$ is 'it is provable that —.' One can then translate a sentential formula δ into the radical formulae δ^* by iterated application of the following translation schemes:

$$\begin{array}{ccc} \mathbb{J}^* = \neg\bot & \mathbb{U}^* = \bot & (\vdash \alpha)^* = \Box\,\alpha \\ (\delta_1 \Cap \delta_2)^* = (\delta_1^* \wedge \delta_2^*) & (\delta_1 \Cup \delta_2)^* = (\delta_1^* \vee \delta_2^*) & (\delta_1 \Supset \delta_2)^* = \Box\,(\delta_1^* \rightarrow \delta_2^*) \end{array}$$

The existence of such an embedding of sentential formulae into the class of radical formulae is called the reflection principle of the logic for pragmatics. The reflection principle enables one to give a so-called descriptive reading to illocutions or speech acts. For instance, the descriptive reading of 'the assertion that —' is 'it is provable that —.' The upshot is that the reflection principle provides means to pin down formally the close relationship between the concepts of assertion and proof introduced by Frege.

The concept of obligation Dalla Pozza concludes his basic account of a logic for pragmatics with some remarks on possible further developments of the conceptual framework. In particular, he proposes that

> [...] the apparatus of logical-pragmatic signs [...] could be enriched by introducing [...] further signs of pragmatic mode, such as [...] deontic modality; consequently the apparatus of the pragmatic formulas could be extended in such a way that also [...] deontic logic (understood as the logic of *normative*, or *prescriptive, sentences* and not as the logic of *norm-descriptive sentences*, or *normative propositions*) can be embodied into the pragmatic language [...] [DG95, p. 104]

And he sets out to do so in [Dal97]. From a syntactic point of view it suffices to extend the class of elementary formulae η with the pragmatic operator $\multimap$ for deontic modality that is interpreted as 'the obligation to —.' Therefore, the extended class of sentential formulae is defined by the following grammar:

$$\eta := \vdash \alpha \mid \multimap \alpha \qquad \delta := \eta \mid \mathbb{J} \mid \mathbb{U} \mid \delta \cap \delta \mid \delta \cup \delta \mid \delta \supset \delta$$

Contrary to the case for assertions one has that an act of obligation or norm $\multimap \alpha$ is always impersonal. The justification of a norm is done in two steps: first, one defines what it means to be justified with respect to some specific normative system $\mathcal{N}$; second, one abstracts away from the particular normative system. By a normative system one simply understands a collection of obligations and permissions where the pragmatic operator $\wp$ (interpreted as 'the permission to —') is defined in terms of $\multimap$ as follows:

$$\wp \alpha =_{def} \sim \multimap (\neg \alpha) =_{def} \multimap (\neg \alpha) \supset \mathbb{U}$$

One has that an act of obligation $\multimap \alpha$ is justified with respect to a normative system $\mathcal{N}$ if and only if there is a proof that (a) $\multimap \alpha$ satisfies the specific conditions for membership in $\mathcal{N}$, (b) α is the description of an act or action, (c) α is physically possible, and (d) α is logically compatible with every α' that occurs in either an obligation $\circ\!\!-\alpha'$ or a permission $\wp\,\alpha'$ of $\mathcal{N}$. In other words, the justification of an obligation or norm with respect to some specific normative system $\mathcal{N}$ is based on four necessary conditions: (a) existence, (b) content, (c) satisfiability, and (d) compatibility. Furthermore, the requirement that there exists a proof showing that these conditions hold introduces a strong criterion of rationality that enables one to recognise a norm that is logically entailed by a set of justified norms as justified. One then has that an act of obligation $\circ\!\!-\alpha$ is justified if and only if it is justified with respect to every normative system. Note that the concept of pragmatic validity carries over without further modifications.

A logic of norms Having extended the framework of a logic of judgements or assertions with the concept of obligation, Dalla Pozza is then able to tackle the fundamental philosophical problem of norms, i.e. the problem of whether there is a logic of norms. Put in simple terms, the nucleus of the problem is that a norm is usually regarded as a prescription to bring or to not bring about a certain state of affairs and as such cannot be endowed with a truth value. In other words, classical logic can be conceived as a logic of norms only if norms are regarded as norm-descriptive rather than as prescriptive sentences. Dalla Pozza thesis is that there is a logic of norms where norms are not regarded as norm-descriptive sentences, namely the logic of judgements endowed with a pragmatic operator for obligation: the argument is essentially based on the fact that the concept of pragmatic validity extends to norms as well. Instead of providing a detailed account of all the philosophical conclusions that follow from the thesis (see for instance [Dal97]) let us just point out how the reflection principle can be extended to the logic of judgements and norms. As above, one considers an extended class of radical formulae α and a translation scheme for the new pragmatic operator:[4]

$$\alpha := p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \Box\alpha \mid \circ\,\alpha \qquad (\circ\!\!-\alpha)^* = \Box \circ \alpha$$

The intended reading of the classical modal operator $\circ$ obeying the laws of **KD** is 'it is obligatory to —' and so the descriptive reading of 'the obligation to —' is 'it is provable that it is obligatory to —.' Here again, the reflection principle exhibits the close relationship that holds between norms and their justification. It is worth mentioning that the elementary formulae $\circ\!\!-\alpha$ and $\vdash(\circ\,\alpha)$ are equivalent with respect to the translation scheme. The equivalence is best explained in terms of the various interpretations that might be given to an obligation sign. Indeed, a 'no smoking'

[4] The translation scheme was actually introduced by Gianluigi Bellin and the present author in [BR03].

sign for example may be rendered either as 'the obligation to not smoke' or as 'the assertion that it is obligatory to not smoke', and both readings are plausible from a common-sense perspective. However, $\multimap \alpha$ denotes a norm and is thus different from $\vdash (\circ\, \alpha)$ which denotes an assertion, i.e. $\multimap \alpha$ cannot be identified with $\vdash (\circ\, \alpha)$.

Some related work We conclude this section with a few pointers to related work where other pragmatic operators or signs of pragmatic mode have been considered within the framework of a logic for pragmatics. Most notably, Bellin and Biasi have investigated the possibility of a logic for pragmatics with assertions and conjectures (see for instance [BB04] and [BA08]). Furthermore, Dalla Pozza has outlined an extension of the basic account dealing with questions in a presentation at the University of Verona in the autumn of 2003.

2.2 Causality and hypothetical norms

Starting from the logic of judgements and norms Bellin and Dalla Pozza investigate whether it is possible to give a pragmatic interpretation of linear logic, thus extending the framework with a concept of causality. We further enhance the framework by introducing a concept of conditional obligation or hypothetical norm.

Notational conventions Before we start with the exposition of the main ideas let us introduce some new notation. We set $\eta_\alpha =_{def} \vdash \alpha$ and $\eta^\circ_\alpha =_{def} \multimap \alpha$. The class of sentential formulae is thus defined as follows:

$$\eta := \eta_\alpha \,|\, \eta^\circ_\alpha \qquad \delta := \eta \,|\, \mathbb{J} \,|\, \mathbb{U} \,|\, \delta \cap \delta \,|\, \delta \cup \delta \,|\, \delta \supset \delta$$

The use of meta-variables instead of pragmatic operators or signs of pragmatic mode has several reasons. First, it simply makes the formal systems that will be considered in this thesis more readable and thus helps to avoid needless confusion. Second, it provides a neat graphical way to symbolise the shift from an assertion with content α to an obligation with content α that we shall see below. Third and most importantly, the $(-)^\circ$ operator can be seen as an abstract operator, thus making it possible to postpone decisions about possible interpretations. As we shall see at the end of this section one can specialise this operator in such a way that hypothetical norms can be interpreted in terms of it.

The concept of causality One of the aims of [BD02] is to provide a well-behaved mixed system that formalises reasoning about assertions and obligations, presented in terms of sequent calculus rules. Of particular interest is a principle that lets interact acts of assertion and obligation via a notion of causal implication. To put it in Anscombe's words, "the topic of causality is in a state of too great confusion" [Ans00, p. 10], and Bellin and Dalla Pozza take great care to make explicit their basic assumptions. First of all, one has that causal links are established only between acts of assertions, thus making it possible to give them a justification in terms of scientific laws. Indeed, if $\Supset$ stands for causal implication then $\eta_{\alpha_1} \Supset \eta_{\alpha_2}$ is justified if and only if there exists some scientific law $\forall x.\varphi(x) \to \psi(x)$ and some term t such that $\alpha_1 = \varphi(t)$ and $\alpha_2 = \psi(t)$. It is worth pointing out that the universal quantifier is used in a semi-formal way: the basic idea is that a scientific law is considered as some general pattern that holds universally and that a causal implication is justified if it can be regarded as an instance of it. The justification of a causal implication is thus extensional, i.e. it depends only on the existence of the scientific law and the justification of η_{α_1}. As an immediate consequence of extensionality one has that causal implication has its intended meaning only in positive occurrences, i.e. in causal formulae of the form $\eta_{\alpha_1} \Supset (\eta_{\alpha_2} \Supset (\ldots \Supset (\eta_{\alpha_n} \Supset \eta_\alpha)\ldots))$. The extended class of sentential formulae is thus defined as follows:

$$\xi := \eta_\alpha \,|\, \eta_\alpha \Supset \xi \qquad \delta := \eta^\circ_\alpha \,|\, \xi \,|\, \mathbb{J} \,|\, \mathbb{U} \,|\, \delta \cap \delta \,|\, \delta \cup \delta \,|\, \delta \supset \delta$$

Note that assertive elementary formulae η_α are absorbed by the new class of pure causal formulae ξ and normative elementary formulae η°_α by the class of sentential formulae δ. One might wonder whether there are any causal links between acts of obligation: these are established indirectly via the sequent calculus rule

$$\frac{\eta_{\alpha_1}, \ldots, \eta_{\alpha_n}, \xi_1, \ldots, \xi_m \Rightarrow_{\mathrm{c}} \eta_\alpha}{\eta^\circ_{\alpha_1}, \ldots, \eta^\circ_{\alpha_n}, \xi_1, \ldots, \xi_m \Rightarrow_{\mathrm{d}} \eta^\circ_\alpha}\ (-)^\circ$$

which we have already encountered in the previous chapter. Another important aspect besides extensionality is that the formulae in the antecedent of the sequent have to be relevant with respect to the formula in the succedent. This is achieved by admitting only the structural rule of contraction but not the weakening rule.

Comments and remarks Since the main theme of this thesis is the study of the $(-)^\circ$ rule from a proof-theoretical and semantic perspective it is worth having a closer look at it. As already mentioned in the introduction the rule can be regarded as a parametrised version of the K rule familiar from modal logic. Note however that, as a consequence of the fact that there is a change of inference relation, one has only a flat modality. This is in accordance with the basic requirement that signs of pragmatic mode cannot be iterated. The purpose of the rule is to transform assertive (elementary) formulae in the antecedent and the succedent of a sequent into normative (elementary) formulae, thus providing means to make a shift from assertions to obligations. Note that in contrast to [BD02] we do not require that each parameter ξ_i is a causal implication, i.e. it might be the case that some parameter ξ_i is an assertive (elementary) formula. This can lead to violations of Hume's law which says that one cannot derive an *ought* from an *is*. Indeed, consider an instance of the above rule where $n = 0$: in this case there is no normative (elementary) formula in the antecedent of the conclusion. We shall return to this important issue when we specialise the $(-)^{\circ}$ operator at the end of this section.

Commutativity and events The attentive reader will have noticed that we have only talked about the presence or not of the structural rules of weakening and contraction in the sequent calculus presentation of the $(-)^{\circ}$ rule. In the light of the difficulties that arise when one considers non-commutative logics Bellin and Dalla Pozza make the simplifying assumption that one admits the structural rule of exchange. Formally, we have that

$$\eta_{\alpha_1} \Supset (\eta_{\alpha_2} \Supset \xi) \sim \eta_{\alpha_2} \Supset (\eta_{\alpha_1} \Supset \xi) \sim (\eta_{\alpha_1} \circ \eta_{\alpha_2}) \Supset \xi$$

where $\sim$ stands for interderivability and $\circ$ for relevant conjunction (see for instance [Res00]). Note that with respect to [BD02] we have thus introduced the connective $\circ$. There are basically two reasons for this choice. First, from a semantic point of view it is desirable to have a so-called left adjoint to causal implication. It allows one to consider the commas in the antecedent of a sequent as conjunctions, thus simplifying the definition of the interpretation of the logic in the semantics. Second, although the informal interpretation of causal implication is based on a reduction to a truth-functional or descriptive condition the concept of causality by itself establishes a link between events. In particular, the interpretation of causality in terms of conditional probabilities (see for instance [Pea00]) is based on the idea that a certain number of distinct events occurring in the same spatiotemporal region are necessary to trigger some other event. The new class of causal-deontic formulae γ is defined as follows:

$$\eta := \eta_\alpha \,|\, \epsilon \,|\, \eta \circ \eta \qquad \xi := \eta \,|\, \eta \Supset \xi \,|\, \xi \circ \xi \qquad \gamma := \eta^{\circ} \,|\, \xi \,|\, \gamma \circ \gamma$$

Note that this is simply a refinement of the above grammar where we have introduced the relevant conjunction $\circ$ and its unit ϵ. It is worth mentioning that the event reading of relevant conjunction applies only to the class of assertive formulae η; relevant conjunctions of causal-deontic formulae where at least one of the subformulae is not an assertive formula η have been mainly introduced for the abovementioned technical reasons.

Relation to linear logic The investigation in [BD02] culminates with a series of translation schemes that aim to show that the linear entailment relation expresses the deductive properties of pragmatic schemes where the pragmatic operators are unknown and their specification is not required to be uniform. Instead of going into the details of the argument we rather focus on the relation between linear implication and causality. Girard's analysis [Gir87, Gir95] of intuitionistic implication in terms of linear implication $A \supset B = !A \multimap B$ becomes implausible if $A \multimap B$ is given a strict causal interpretation. Although linear implication may sometimes admit a causal interpretation, it is more appropriately regarded as an abstract construction. Indeed, in accordance with the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic mentioned above a justification of $\eta_1 \supset \eta_2$ is provided by a method that transforms a proof of η_1 into a proof of η_2; but it would seem odd to claim that a causal component can always be identified within such a method. However, the informal interpretation of causal implication given above and hence the syntactic restrictions on causal implication imposed by it allow one to show that there is a close connection between causal implications and so-called Horn implications, i.e. linear logic formulae of the form $(p_1 \otimes \cdots \otimes p_n) \multimap (q_1 \otimes \cdots \otimes q_m)$ where p_i and q_j are atomic propositions (see for instance [Kan94]). Since the formal argument requires a bit of proof theory it is postponed until the next chapter (see remark 3.1.4).

On conditional obligation As mentioned at the beginning of this section we shall now see how the $(-)^\circ$ operator can be specialised in such a way that hypothetical norms can be interpreted in terms of it. The key point is to observe that this can be done by introducing a new grammar for η°. As pointed out in [Dal97] one is interested in a new connective $/\!/$ that provides means to relate obligations to assertions is such a way that $\eta_2^\circ /\!/ \eta_1$ supports the following informal interpretation: η_2° is justified conditionally upon the fact that η_1 is justified. However, there is a significant difference between assertions and obligations that needs to be taken into account, namely that the justification of an assertion does not explicitly depend on its content whereas the justification of an obligation does. Remember that the content condition is one of the four conditions listed in the definition of justification of an obligation or norm with respect to some specific normative system $\mathcal{N}$: it says that the content α of an obligation can only be the description of an act or action. Therefore one can safely introduce the restriction that a normative elementary formula η_α° is well-defined only if α satisfies the content condition, i.e. only if α describes an act or action. In order to be able to express this restriction in a succinct way we use $\mathcal{Q}$ to denote the set of radical formulae α that satisfy the content condition. The extended class of causal-deontic formulae is then defined as follows:

$$\gamma := \eta^\circ \,|\, \xi \,|\, \gamma \circ \gamma \quad \text{where} \quad \eta^\circ := \eta_\alpha^\circ \,|\, \eta^\circ /\!\!/ \eta \,|\, \eta^\circ \circ \eta^\circ \quad \text{and} \quad \alpha \in \mathcal{Q}$$

If we use $rad(\eta)$ to denote the set of radical formulae occurring in an assertive formula η then we have that the $(-)^\circ$ operator can only be applied to assertive formulae η where $rad(\eta)$ is a non-empty subset of $\mathcal{Q}$. Thus, it cannot be applied to the assertive formulae ϵ. Since in the semantics considered below the $(-)^\circ$ operator is interpreted as a map from the interpretation of an assertive formula η to the interpretation of the normative formula η° we have that one cannot interpret the conclusion of an $(-)^\circ$ rule

$$\frac{\eta_1,\ldots,\eta_n,\xi_1,\ldots,\xi_m \Rightarrow_{\mathrm{c}} \eta}{\eta_1^\circ,\ldots,\eta_n^\circ,\xi_1,\ldots,\xi_m \Rightarrow_{\mathrm{d}} \eta^\circ}\,(-)^\circ$$

when $n = 0$, the reason being simply that one would have to apply the $(-)^\circ$ operator to ϵ. In other words, the grammar for η° naturally reflects the fact that Hume's law cannot be violated. Furthermore, since $rad(\eta_1 \circ \eta_2) = rad(\eta_1) \cup rad(\eta_2)$, we have that $(\eta_1 \circ \eta_2)^\circ$ is well-defined whenever both η_1° and η_2° are. We can thus make the simplifying assumption that the $(-)^\circ$ operator distributes over the relevant conjunction $\circ$, i.e. that $(\eta_1 \circ \eta_2)^\circ \sim \eta_1^\circ \circ \eta_2^\circ$ where, as above, $\sim$ stands for interderivability.

2.3 Definition of pragmatic language

To sum up, we conclude this chapter with the formal definition of pragmatic language. One should note that we focus on the so-called intuitionistic fragment where the radical part α consists only of atomic propositions p. As above, we use $\mathcal{Q}$ to denote the set of atomic propositions that may be the content of an obligation.

Definition 2.3.1 The *full intuitionistic* fragment $\mathcal{L}_F^P$ of the pragmatic language is defined by the following grammar:

1. Assertive formulae $\eta := \eta_p \,|\, \epsilon \,|\, \eta \circ \eta$;

2. Pure causal formulae $\xi := \eta \mid \eta \supset \xi \mid \xi \circ \xi$;

3. Causal-deontic formulae $\gamma := \eta^\circ \mid \xi \mid \gamma \circ \gamma$;

4. Sentential formulae $\delta := \gamma \mid \mathbb{J} \mid \mathbb{U} \mid \delta \cap \delta \mid \delta \cup \delta \mid \delta \supset \delta$.

We shall use $\mathcal{L}_B^P$ to denote the *basic causal-deontic* fragment of $\mathcal{L}_F^P$, i.e. the fragment of $\mathcal{L}_F^P$ that does not contain sentential formulae δ. Further, we shall use $\mathcal{L}_E^P$ to denote the *extended causal-deontic*, i.e. the basic causal-deontic fragment $\mathcal{L}_B^P$ of $\mathcal{L}_F^P$ with normative formulae η° defined by the grammar $\eta_p^\circ \mid \eta^\circ /\!/ \eta \mid \eta^\circ \circ \eta^\circ$ where p ranges over the set $\mathcal{Q}$.

Remark 2.3.2 Since we deal with radical formulae α of the form p only, i.e. neglect the fact that propositions α actually obey the laws of classical logic, we have to do with an intuitionstic framework. Let us point out that this is also the reason why we call the language defined by the above grammar the full intuitionistic fragment of the pragmatic language.

Remark 2.3.3 We shall use $\vec{\eta}$, Ξ, Γ, and Δ to denote multisets of assertive, pure causal, causal-deontic, and sentential formulae, respectively. As a consequence of the definition of pragmatic language, one has that $\vec{\eta} \subseteq \Xi \subseteq \Gamma \subseteq \Delta$ where $\subseteq$ stands for multiset inclusion. Furthermore, given $\vec{\eta} = \eta_1, \ldots, \eta_n$, we use $(\vec{\eta})^\circ$ as shorthand notation for $\eta_1^\circ, \ldots \eta_n^\circ$.

Chapter 3

Proof-theoretical aspects

We start our technical investigation with a sequent calculus system for the basic causal-deontic fragment of the pragmatic language defined in the previous chapter, one of the reasons being that a sequent calculus system for the logic for pragmatics has also been given in [BD02]. It is worth pointing out, however, that our presentation of the rules differs in many ways from the one provided by Bellin and Dalla Pozza and that a comparison of the two systems is postponed to chapter 8. The main aim of this chapter is to introduce the rules of the system and to show that the system satisfies the cut elimination property. As we shall see in chapter 5, the reductions used in the proof of cut elimination provide a rationale for deciding whether two derivations are essentially the same. It is worth reminding the reader that in this and the next three chapters we deal only with the basic causal-deontic system, the treatment of the extended causal-deontic and full intuitionistic systems being postponed to chapters 7 and 8.

3.1 The basic causal-deontic system

The key feature of the sequent calculus system provided in this section is that it is split into three fragments, each corresponding to one of the layers of the basic causal-deontic fragment $\mathcal{L}_B^P$ of the pragmatic language given in definition 2.3.1 and each being characterised by one of the entailment relations $\Rightarrow_{\mathbf{a}}$, $\Rightarrow_{\mathbf{c}}$, and $\Rightarrow_{\mathbf{d}}$. The use of several entailment relations is inspired by the work on linear and non-linear logic in [Ben94, MMdPR05] and the work on the logic of linear functors in [BCS02]. The basic causal-deontic rules are provided in figure 3.1. Note that most of these correspond to standard rules of the multiplicative fragment of intuitionistic linear logic

$$\frac{}{\eta_p \Rightarrow_{\mathtt{a}} \eta_p}\,\mathrm{ax} \qquad \frac{}{\Rightarrow_{\mathtt{a}} \epsilon}\,\mathcal{R}\epsilon$$

$$\frac{\vec{\eta} \Rightarrow_{\mathtt{a}} \eta}{\epsilon, \vec{\eta} \Rightarrow_{\mathtt{a}} \eta}\,\mathcal{L}\epsilon \qquad \frac{\vec{\eta} \Rightarrow_{\mathtt{a}} \eta}{\vec{\eta} \Rightarrow_{\mathtt{c}} \eta}\,(\mathtt{a/c})$$

$$\frac{\Xi_1 \Rightarrow_{\mathtt{c}} \eta \quad \xi, \Xi_2 \Rightarrow_{\mathtt{c}} \xi'}{\eta \supset \xi, \Xi_1, \Xi_2 \Rightarrow_{\mathtt{c}} \xi'}\,\mathcal{L}\supset \qquad \frac{\Xi, \eta \Rightarrow_{\mathtt{c}} \xi}{\Xi \Rightarrow_{\mathtt{c}} \eta \supset \xi}\,\mathcal{R}\supset$$

$$\frac{\Xi \Rightarrow_{\mathtt{c}} \xi}{\Xi \Rightarrow_{\mathtt{d}} \xi}\,(\mathtt{c/d}) \qquad \frac{\vec{\eta}, \Xi \Rightarrow_{\mathtt{c}} \eta}{(\vec{\eta})^{\circ}, \Xi \Rightarrow_{\mathtt{d}} \eta^{\circ}}\,(-)^{\circ}$$

$$\frac{\gamma', \gamma', \Gamma \Rightarrow_{\mathtt{x}} \gamma}{\gamma', \Gamma \Rightarrow_{\mathtt{x}} \gamma}\,\mathtt{con} \qquad \frac{\Gamma_1 \Rightarrow_{\mathtt{x}} \gamma' \quad \gamma', \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma}{\Gamma_1, \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma}\,\mathtt{cut}$$

$$\frac{\gamma_1, \gamma_2, \Gamma \Rightarrow_{\mathtt{x}} \gamma}{\gamma_1 \circ \gamma_2, \Gamma \Rightarrow_{\mathtt{x}} \gamma}\,\mathcal{L}\circ \qquad \frac{\Gamma_1 \Rightarrow_{\mathtt{x}} \gamma_1 \quad \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma_2}{\Gamma_1, \Gamma_2 \Rightarrow_{\mathtt{x}} \gamma_1 \circ \gamma_2}\,\mathcal{R}\circ$$

Figure 3.1: Basic causal-deontic rules

(see for instance [Bie94]) endowed with the contraction rule.

Remark 3.1.1 In order to avoid writing down several instances of analogous cases, we find it convenient to use the notation $\Gamma \Rightarrow_{\mathtt{x}} \gamma$ for sequents with an indeterminate entailment relation, the particular instance of $\Rightarrow_{\mathtt{x}}$ being determined by the formulae in the antecedent and the consequent of the sequent. Note that, without further mention, we shall use this notational convention also for other formalisations of the basic-causal deontic system.

Remark 3.1.2 Both the (a/c) and (c/d) rules serve the purpose to embed certain fragments of the basic causal-deontic system into other fragments of the system. In particular, we have that

1. the assertive fragment is included in the pure causal fragment;

2. the pure causal fragment is included in the causal-deontic fragment.

This feature of the logic is a direct consequence of the multiset inclusion $\vec{\eta} \subseteq \Xi \subseteq \Gamma$ mentioned at the end of the previous chapter and can succinctly be expressed as $\Rightarrow_{\mathtt{a}} \subseteq \Rightarrow_{\mathtt{c}} \subseteq \Rightarrow_{\mathtt{d}}$, a series of inclusions of entailment relations.

Remark 3.1.3 In the $\mathcal{L}\Supset$ and $\mathcal{R}\Supset$ rules the antecedent of the causal implication is restricted to assertive formulae only. As pointed out in chapter 2 the reason for such a choice is that we want the justification of a causal implication to be extensional.

Remark 3.1.4 The pure causal system, i.e. the fragment without sequents of the form $\Gamma \Rightarrow_{\mathsf{d}} \gamma$, can be related to the Horn fragment of linear logic (see for instance [Kan94]) dealing only with the connectives $\otimes$ and $\multimap$. On the one hand we have that relevant conjunctions may 'absorb' causal implications:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \eta_1 \Rightarrow_{\mathsf{c}} \eta_1\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \eta_2 \Rightarrow_{\mathsf{c}} \eta_2\end{array} \qquad \begin{array}{c}\vdots\\ \xi \Rightarrow_{\mathsf{c}} \xi\end{array}}{\eta_2, \eta_2 \Supset \xi \Rightarrow_{\mathsf{c}} \xi}\,\mathcal{L}\Supset}{\eta_1, \eta_2, \eta_1 \Supset (\eta_2 \Supset \xi) \Rightarrow_{\mathsf{c}} \xi}\,\mathcal{L}\Supset}{\eta_1 \circ \eta_2, \eta_1 \Supset (\eta_2 \Supset \xi) \Rightarrow_{\mathsf{c}} \xi}\,\mathcal{L}\circ}{\eta_1 \Supset (\eta_2 \Supset \xi) \Rightarrow_{\mathsf{c}} (\eta_1 \circ \eta_2) \Supset \xi}\,\mathcal{R}\Supset
$$

On the other hand we have that relevant conjunctions may 'distribute' over causal implications:[1]

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \eta_1 \Rightarrow_{\mathsf{c}} \eta_1\end{array} \qquad \begin{array}{c}\vdots\\ \xi_1 \Rightarrow_{\mathsf{c}} \xi_1\end{array}}{\eta_1, \eta_1 \Supset \xi_1 \Rightarrow_{\mathsf{c}} \xi_1}\,\mathcal{L}\Supset \qquad \dfrac{\begin{array}{c}\vdots\\ \eta_2 \Rightarrow_{\mathsf{c}} \eta_2\end{array} \qquad \begin{array}{c}\vdots\\ \xi_2 \Rightarrow_{\mathsf{c}} \xi_2\end{array}}{\eta_2, \eta_2 \Supset \xi_2 \Rightarrow_{\mathsf{c}} \xi_2}\,\mathcal{L}\Supset}{\eta_1, \eta_2, \eta_1 \Supset \xi_1, \eta_2 \Supset \xi_2 \Rightarrow_{\mathsf{c}} \xi_1 \circ \xi_2}\,\mathcal{R}\circ}{\overline{\eta_1 \circ \eta_2, (\eta_1 \Supset \xi_1) \circ (\eta_2 \Supset \xi_2) \Rightarrow_{\mathsf{c}} \xi_1 \circ \xi_2}}\,\mathcal{L}\circ}{(\eta_1 \Supset \xi_1) \circ (\eta_2 \Supset \xi_2) \Rightarrow_{\mathsf{c}} (\eta_1 \circ \eta_2) \Supset (\xi_1 \circ \xi_2)}\,\mathcal{R}\Supset
$$

These two properties taken together allow one to derive a sequent $\mathcal{S}_\xi = (\xi \Rightarrow_{\mathsf{c}} (\eta_\xi^-) \Supset (\eta_\xi^+))$ for each pure causal formula ξ where η_ξ^- (resp. η_ξ^+) denotes the relevant conjunction of all the assertive formulae η with negative (resp. positive) occurrence in ξ; the straightforward proof by induction on the complexity of ξ is omitted. As a consequence, a derivation of the sequent $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_m \Rightarrow_{\mathsf{c}} \eta$ where each ξ_i contains at least one causal implication can be transformed into a derivation

$$
\dfrac{\overline{\mathcal{S}_{\xi_1} \quad \cdots \quad \mathcal{S}_{\xi_m} \qquad \begin{array}{c}(\mathrm{HLL} + \mathtt{con})\\ \eta_1 \circ \cdots \circ \eta_n, \eta_{\xi_1}^- \Supset \eta_{\xi_1}^+, \ldots, \eta_{\xi_m}^- \Supset \eta_{\xi_m}^+ \Rightarrow_{\mathsf{c}} \eta\end{array}}}{\eta_1 \circ \cdots \circ \eta_n, \xi_1, \ldots, \xi_m \Rightarrow_{\mathsf{c}} \eta}\,\mathtt{cut}
$$

[1]Throughout the thesis we shall use the convention that whenever there are several consecutive instances of some rule in a derivation we indicate it by using a double line instead of the usual single line.

in Horn linear logic **HLL** (see [Kan94, table 3 on p. 204]) extended with an axiom scheme $\mathcal{S}_\xi$ and the structural rule of contraction. Note that restricting the consequent of the sequent to an assertive formula can be done without loss of generality: it is straightforward to show that if $\Xi \Rightarrow_{\mathtt{c}} \xi$ is derivable in sequent calculus then so is $\eta_\xi^-, \Xi \Rightarrow_{\mathtt{c}} \eta_\xi^+$.

Remark 3.1.5 Of crucial importance is the $(-)^\circ$ rule already seen in the previous chapters: it formalises the propagation of obligations or norms through causal chains of assertions. Note that, analogously to the $(\mathtt{c}/\mathtt{d})$ rule, it transforms a sequent of the pure causal fragment of the system into a sequent of the causal-deontic fragment.

Definition 3.1.6 The *depth* $|\gamma|$ *of a causal-deontic formula* γ is defined inductively as follows:

1. $|\eta_p| = |\epsilon| = 0$;
2. $|\eta^\circ| = |\eta| + 1$;
3. $|\eta \supset\!\!\!\!\supset \xi| = \max(|\eta|, |\xi|) + 1$;
4. $|\gamma_1 \circ \gamma_2| = \max(|\gamma_1|, |\gamma_2|) + 1$.

Lemma 3.1.7 The following generalised rules

$$\frac{}{\gamma \Rightarrow_{\mathtt{x}} \gamma}\,\mathrm{ax} \qquad\qquad \frac{}{\Rightarrow_{\mathtt{x}} \epsilon}\,\mathcal{R}\epsilon \qquad\qquad \frac{\Gamma \Rightarrow_{\mathtt{x}} \gamma}{\epsilon, \Gamma \Rightarrow_{\mathtt{x}} \gamma}\,\mathcal{L}\epsilon$$

are admissible in the basic causal-deontic system.

Proof We consider only an instance of the generalised axiom rule. Given that $\prec$ denotes the standard ascending order on the letters of the alphabet, the proof is then by induction on the ordered pair $(|\gamma|, x)$ where x ranges over the set $\{\mathtt{a}, \mathtt{c}, \mathtt{d}\}$. The most interesting cases are the following ones:

1. The axiom

$$\frac{}{\eta \Rightarrow_{\mathtt{c}} \eta}\,\mathrm{ax}$$

is replaced by

$$\frac{\dfrac{}{\eta \Rightarrow_{\mathtt{a}} \eta}\,\mathrm{ax}}{\eta \Rightarrow_{\mathtt{c}} \eta}\,(\mathtt{a}/\mathtt{c})$$

and, since $\mathtt{a} \prec \mathtt{c}$, the induction hypothesis applies.

2. The axiom

$$\frac{}{\xi \Rightarrow_{\mathsf{d}} \xi}\,\mathrm{ax}$$

is replaced by

$$\frac{\dfrac{}{\xi \Rightarrow_{\mathsf{c}} \xi}\,\mathrm{ax}}{\xi \Rightarrow_{\mathsf{d}} \xi}\,(\mathsf{c}/\mathsf{d})$$

and, since $\mathsf{c} \prec \mathsf{d}$, the induction hypothesis applies.

3. The axiom

$$\frac{}{\eta^{\circ} \Rightarrow_{\mathsf{d}} \eta^{\circ}}\,\mathrm{ax}$$

is replaced by

$$\frac{\dfrac{}{\eta \Rightarrow_{\mathsf{c}} \eta}\,\mathrm{ax}}{\eta^{\circ} \Rightarrow_{\mathsf{d}} \eta^{\circ}}\,(-)^{\circ}$$

and, since $|\eta| < |\eta^{\circ}|$, the induction hypothesis applies. □

Theorem 3.1.8 (cut elimination) If a sequent $\Gamma \Rightarrow_{\mathsf{x}} \gamma$ is derivable according to the rules of figure 3.1 then it has a cut-free derivation.

3.2 Proof of cut elimination theorem

If the derivation of $\Gamma \Rightarrow_{\mathsf{x}} \gamma$ in theorem 3.1.8 is not already cut-free, then the proof proceeds by iterated application of lemma 3.2.3 below to the top-most cut in the prooftree, thus reducing the number of cuts by one at each iteration.

Definition 3.2.1 The *rank* of a cut on a causal-deontic formula γ is the ordered pair $(|\gamma| + 1, x)$ where x ranges over the set of letters $\{\mathsf{a}, \mathsf{c}, \mathsf{d}\}$ and $\prec$ denotes the standard ascending order on the letters of the alphabet.

Definition 3.2.2 The *level* of a cut is the sum of the heights of the derivations of the premises where the height of a derivation is the maximum length of the branches in the prooftree and the length of a branch is the number of nodes in the branch minus 1.

Lemma 3.2.3 If π_1 and π_2 in

$$\frac{\begin{matrix}\pi_1 \\ \Gamma_1 \Rightarrow_{\mathsf{x}} \gamma'\end{matrix} \qquad \begin{matrix}\pi_2 \\ \gamma', \Gamma_2 \Rightarrow_{\mathsf{x}} \gamma\end{matrix}}{\Gamma_1, \Gamma_2 \Rightarrow_{\mathsf{x}} \gamma}\,\mathrm{cut}$$

are cut-free derivations of $\Gamma_1 \Rightarrow_x \gamma'$ and $\gamma', \Gamma_2 \Rightarrow_x \gamma$, respectively, then there exists a cut-free derivation of $\Gamma_1, \Gamma_2 \Rightarrow_x \gamma$.

Proof The proof is by induction on the ordered pair (r, l) where r and l are the rank of the cut on γ and the level of the cut, respectively. The basic idea is to replace a cut with simpler cuts, i.e with cuts that have either a smaller rank than, or the same rank but a lower level than, the original one. However, in the presence of the structural rule of contraction this is not always possible since the cut in

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma_1 \Rightarrow_x \gamma'\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \gamma', \gamma', \Gamma_2 \Rightarrow_x \gamma\end{array}}{\gamma', \Gamma_2 \Rightarrow_x \gamma}\,\texttt{con}}{\Gamma_1, \Gamma_2 \Rightarrow_x \gamma}\,\texttt{cut}$$

should be replaced by two cuts

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma_1 \Rightarrow_x \gamma'\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \Gamma_1 \Rightarrow_x \gamma'\end{array} \qquad \begin{array}{c}\pi_2\\ \gamma', \gamma', \Gamma_2 \Rightarrow_x \gamma\end{array}}{\gamma', \Gamma_1, \Gamma_2 \Rightarrow_x \gamma}\,\texttt{cut}}{\Gamma_1, \Gamma_1, \Gamma_2 \Rightarrow_x \gamma}\,\texttt{cut}^{\dagger}}{\Gamma_1, \Gamma_2 \Rightarrow_x \gamma}\,\texttt{con}$$

but the level of the cut labelled with $\dagger$ is not necessarily lower than the level of the original cut. The simplest solution to this problem is to use multicuts as in Gentzen's original proof [Gen35]. The multicut rule

$$\dfrac{\Gamma_1 \Rightarrow_x \gamma' \qquad \gamma', \ldots, \gamma', \Gamma_2 \Rightarrow_x \gamma}{\Gamma_1, \Gamma_2 \Rightarrow_x \gamma}\,\texttt{cut}$$

allows several copies of the cut formula γ' in the right premise of the rule. It is straightforward to show that the multicut rule is admissible in the system and hence it can be used instead of a simple cut whenever necessary. Now, there are the following three cases to consider.

1. Either the left or the right premise of a cut is an axiom. In the latter case we have that

$$\dfrac{\begin{array}{c}\pi\\ \vec{\eta} \Rightarrow_a \eta_p\end{array} \qquad \dfrac{}{\eta_p \Rightarrow_a \eta_p}\,\texttt{ax}}{\vec{\eta} \Rightarrow_a \eta_p}\,\texttt{cut}$$

reduces to the cut-free derivation

$$\begin{array}{c}\pi\\ \vec{\eta} \Rightarrow_a \eta_p\end{array}$$

The former case is similar and therefore omitted. These two cases form the base case of the induction.

2. If the cut formula is not principal in at least one of the premises of a cut then we permute the order of the rules in the derivation. We consider only one case in detail, all other cases being similar and therefore omitted. If the right premise of the cut is the conclusion of an instance of $\mathcal{R}\circ$ then either

$$\dfrac{\begin{matrix}\pi\\ \Gamma \Rightarrow_x \gamma\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1\\ \gamma, \Gamma_1 \Rightarrow_x \gamma_1\end{matrix} \qquad \begin{matrix}\pi_2\\ \Gamma_2 \Rightarrow_x \gamma_2\end{matrix}}{\gamma, \Gamma_1, \Gamma_2 \Rightarrow_x \gamma_1 \circ \gamma_2}\,\mathcal{R}\circ}{\Gamma, \Gamma_1, \Gamma_2 \Rightarrow_x \gamma_1 \circ \gamma_2}\,\mathrm{cut}$$

reduces to

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \Gamma \Rightarrow_x \gamma\end{matrix} \qquad \begin{matrix}\pi_1\\ \gamma, \Gamma_1 \Rightarrow_x \gamma_1\end{matrix}}{\Gamma, \Gamma_1 \Rightarrow_x \gamma_1}\,\mathrm{cut} \qquad \begin{matrix}\pi_2\\ \Gamma_2 \Rightarrow_x \gamma_2\end{matrix}}{\Gamma, \Gamma_1, \Gamma_2 \Rightarrow_x \gamma_1 \circ \gamma_2}\,\mathcal{R}\circ$$

or

$$\dfrac{\begin{matrix}\pi\\ \Gamma \Rightarrow_x \gamma\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1\\ \Gamma_1 \Rightarrow_x \gamma_1\end{matrix} \qquad \begin{matrix}\pi_2\\ \gamma, \Gamma_2 \Rightarrow_x \gamma_2\end{matrix}}{\Gamma_1, \gamma, \Gamma_2 \Rightarrow_x \gamma_1 \circ \gamma_2}\,\mathcal{R}\circ}{\Gamma_1, \Gamma, \Gamma_2 \Rightarrow_x \gamma_1 \circ \gamma_2}\,\mathrm{cut}$$

reduces to

$$\dfrac{\begin{matrix}\pi_1\\ \Gamma_1 \Rightarrow_x \gamma_1\end{matrix} \qquad \dfrac{\begin{matrix}\pi\\ \Gamma \Rightarrow_x \gamma\end{matrix} \qquad \begin{matrix}\pi_2\\ \gamma, \Gamma_2 \Rightarrow_x \gamma_2\end{matrix}}{\Gamma, \Gamma_2 \Rightarrow_x \gamma_2}\,\mathrm{cut}}{\Gamma_1, \Gamma, \Gamma_2 \Rightarrow_x \gamma_1 \circ \gamma_2}\,\mathcal{R}\circ$$

Since in both cases the new cut has the same rank but a lower level than the original one we can apply the induction hypothesis.

3. If the cut formula is principal in both the premises of a cut then we replace the cut with a simpler one, i.e. with a cut of smaller rank. We consider only the cases involving the (a/c), (c/d), and $(-)^\circ$ rules, all other cases being similar and therefore omitted.

   (a) If both premises of the cut are the conclusion of instances of (a/c) then

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vec{\eta}_1 \Rightarrow_a \eta'\end{matrix}}{\vec{\eta}_1 \Rightarrow_c \eta'}\,(\mathrm{a/c}) \qquad \dfrac{\begin{matrix}\pi_2\\ \eta', \vec{\eta}_2 \Rightarrow_a \eta\end{matrix}}{\eta', \vec{\eta}_2 \Rightarrow_c \eta}\,(\mathrm{a/c})}{\vec{\eta}_1, \vec{\eta}_2 \Rightarrow_c \eta}\,\mathrm{cut}$$

reduces to

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vec{\eta}_1 \Rightarrow_{\mathtt{a}} \eta'\end{array} \qquad \begin{array}{c}\pi_2\\ \eta', \vec{\eta}_2 \Rightarrow_{\mathtt{a}} \eta\end{array}}{\vec{\eta}_1, \vec{\eta}_2 \Rightarrow_{\mathtt{a}} \eta}\,\mathtt{cut}}{\vec{\eta}_1, \vec{\eta}_2 \Rightarrow_{\mathtt{c}} \eta}\,(\mathtt{a}/\mathtt{c})$$

Since the new cut has a smaller rank than the original one ($\mathtt{a} \prec \mathtt{c}$) we can apply the induction hypothesis.

(b) If both premises of the cut are the conclusion of instances of $(\mathtt{c}/\mathtt{d})$ then

$$\dfrac{\begin{array}{c}\pi_1\\ \dfrac{\Xi_1 \Rightarrow_{\mathtt{c}} \xi'}{\Xi_1 \Rightarrow_{\mathtt{d}} \xi'}\,(\mathtt{c}/\mathtt{d})\end{array} \qquad \begin{array}{c}\pi_2\\ \dfrac{\xi', \Xi_2 \Rightarrow_{\mathtt{c}} \xi}{\xi', \Xi_2 \Rightarrow_{\mathtt{d}} \xi}\,(\mathtt{c}/\mathtt{d})\end{array}}{\Xi_1, \Xi_2 \Rightarrow_{\mathtt{d}} \xi}\,\mathtt{cut}$$

reduces to

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Xi_1 \Rightarrow_{\mathtt{c}} \xi'\end{array} \qquad \begin{array}{c}\pi_2\\ \xi', \Xi_2 \Rightarrow_{\mathtt{c}} \xi\end{array}}{\Xi_1, \Xi_2 \Rightarrow_{\mathtt{c}} \xi}\,\mathtt{cut}}{\Xi_1, \Xi_2 \Rightarrow_{\mathtt{d}} \xi}\,(\mathtt{c}/\mathtt{d})$$

Since the new cut has a smaller rank than the original one ($\mathtt{c} \prec \mathtt{d}$) we can apply the induction hypothesis.

(c) If the left premise of the cut is the conclusion of an instance of $(\mathtt{c}/\mathtt{d})$ and the right premise is the conclusion of an instance of $(-)^\circ$ then

$$\dfrac{\begin{array}{c}\pi_1\\ \dfrac{\Xi_1 \Rightarrow_{\mathtt{c}} \xi'}{\Xi_1 \Rightarrow_{\mathtt{d}} \xi'}\,(\mathtt{c}/\mathtt{d})\end{array} \qquad \begin{array}{c}\pi_2\\ \dfrac{\vec{\eta}, \xi', \Xi_2 \Rightarrow_{\mathtt{c}} \eta}{(\vec{\eta})^\circ, \xi', \Xi_2 \Rightarrow_{\mathtt{d}} \eta^\circ}\,(-)^\circ\end{array}}{(\vec{\eta})^\circ, \Xi_1, \Xi_2 \Rightarrow_{\mathtt{d}} \eta^\circ}\,\mathtt{cut}$$

reduces to

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Xi_1 \Rightarrow_{\mathtt{c}} \xi'\end{array} \qquad \begin{array}{c}\pi_2\\ \vec{\eta}, \xi', \Xi_2 \Rightarrow_{\mathtt{c}} \eta\end{array}}{\vec{\eta}, \Xi_1, \Xi_2 \Rightarrow_{\mathtt{c}} \eta}\,\mathtt{cut}}{(\vec{\eta})^\circ, \Xi_1, \Xi_2 \Rightarrow_{\mathtt{d}} \eta^\circ}\,(-)^\circ$$

Since the new cut has a smaller rank than the original one ($\mathtt{c} \prec \mathtt{d}$) we can apply the induction hypothesis.

(d) If both premises of the cut are the conclusion of an instances of $(-)^{\circ}$ then

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vec{\eta}_1, \Xi_1 \Rightarrow_{\mathsf{c}} \eta'\end{array}}{(\vec{\eta}_1)^{\circ}, \Xi_1 \Rightarrow_{\mathsf{d}} (\eta')^{\circ}}\,(-)^{\circ} \qquad \dfrac{\begin{array}{c}\pi_2\\ \eta', \vec{\eta}_2, \Xi_2 \Rightarrow_{\mathsf{c}} \eta\end{array}}{(\eta')^{\circ}, (\vec{\eta}_2)^{\circ}, \Xi_2 \Rightarrow_{\mathsf{d}} \eta^{\circ}}\,(-)^{\circ}}{(\vec{\eta}_1)^{\circ}, (\vec{\eta}_2)^{\circ}, \Xi_1, \Xi_2 \Rightarrow_{\mathsf{d}} \eta^{\circ}}\,\mathsf{cut}$$

reduces to

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vec{\eta}_1, \Xi_1 \Rightarrow_{\mathsf{c}} \eta'\end{array} \qquad \begin{array}{c}\pi_2\\ \eta', \vec{\eta}_2, \Xi_2 \Rightarrow_{\mathsf{c}} \eta\end{array}}{\vec{\eta}_1, \vec{\eta}_2, \Xi_1, \Xi_2 \Rightarrow_{\mathsf{c}} \eta}\,\mathsf{cut}}{(\vec{\eta}_1)^{\circ}, (\vec{\eta}_2)^{\circ}, \Xi_1, \Xi_2 \Rightarrow_{\mathsf{d}} \eta^{\circ}}\,(-)^{\circ}$$

Since the new cut has a smaller rank than the original one ($|\eta| < |\eta^{\circ}|$) we can apply the induction hypothesis. □

3.3 Discussion and final remarks

We have presented a sequent calculus system for the basic causal-deontic fragment of the pragmatic language and shown that it satisfies the cut elimination property. Proof theory will also play a major role in chapter 5. However, in contrast to this chapter, there we have to deal with a natural deduction system. Let us only mention at this point that the natural deduction presentation of the basic causal-deontic rules given in figure 5.1 and the sequent calculus presentation given in this chapter are equivalent up to provability.

Chapter 4

On a provability semantics

In general we are interested in a semantics that does not rely on the reflection principle discussed in chapter 2, i.e. on a translation of the logic for pragmatics into bimodal logic. We shall now investigate a class of sound and complete Kripke models for the basic causal-deontic fragment that, unlike the ones considered in [BR03], do not rely on such a translation; rather they can be seen as a refinement of the Kripke models for the multiplicative fragment of the logic of bunched implications given in [POY04]. As we shall see in chapter 6, Kripke models give rise to algebraic models, a class of categorical models that deal only with provability.

4.1 Basic Kripke frame and model

Following the standard way of introducing Kripke semantics we first provide the concept of Kripke frame (see definition 4.1.3) and then endow it with a forcing relation in order to get a Kripke model (see definition 4.1.4). Note that the Kripke semantics is defined in such a way that the so-called monotonicity property is satisfied (see lemma 4.1.6).

Definition 4.1.1 A *preordered monoid* $(M, \cdot, 1, \leq)$ is a commutative monoid $(M, \cdot, 1)$ endowed with a preorder $\leq$ such that, for all $m_1, m_1', m_2, m_2' \in M$, if $m_1 \leq m_1'$ and $m_2 \leq m_2'$ then $m_1 \cdot m_2 \leq m_1' \cdot m_2'$.

Definition 4.1.2 A preordered monoid $(M, \cdot, 1, \leq)$ is called *relevant* if, for all $m \in M$, $m \leq m \cdot m$. For the sake of conciseness we shall use relevant monoid instead of relevant preordered monoid throughout this thesis.

Definition 4.1.3 Let $U = (U, \cdot, 1, \preceq)$ and $W = (W, \cdot, 1, \preceq)$ be relevant monoids such that U is a relevant submonoid of W. A *basic Kripke frame* is a triple $(W, U, \lhd)$ where $\lhd \subseteq U \times W$ is a binary relation such that

1. for all $u \in U$ and $w \in W$, if $u \lhd w$ and $w \preceq 1$ then $u \preceq 1$;
2. for all $u \in U$ and $w, w', w'' \in W$, if $u \lhd w$ and $w \preceq w' \cdot w''$ then there exist $u', u'' \in U$ such that $u' \lhd w'$, $u'' \lhd w''$, and $u \preceq u' \cdot u''$;
3. for all $u' \in U$ and $w, w', w'' \in W$, if $u' \lhd w'$ and $w \preceq w' \cdot w''$ then, for all $u \in U$, if $u \lhd w$ then $u \preceq u' \cdot w''$.

Definition 4.1.4 A *basic Kripke model* $(W, U, \lhd, \Vdash)$ is a basic Kripke frame $(W, U, \lhd)$ endowed with a forcing relation $\Vdash \subseteq W \times \Gamma$. Given a downward closed subset[1] $\eta_p^\star$ of U for each propositional atom p, the forcing relation is defined as follows:

1. $w \Vdash \eta_p$ iff $w \in \eta_p^\star$;
2. $w \Vdash \epsilon$ iff $w \preceq 1$;
3. $w \Vdash \eta^\circ$ iff for all $u \in U$ such that $u \lhd w$, $u \Vdash \eta$;
4. $w \Vdash \eta \supset \xi$ iff for all $w' \in W$ such that $w' \Vdash \eta$, $w \cdot w' \Vdash \xi$;
5. $w \Vdash \gamma_1 \circ \gamma_2$ iff there exist $w_1, w_2 \in W$ such that $w \preceq w_1 \cdot w_2$, $w_1 \Vdash \gamma_1$, and $w_2 \Vdash \gamma_2$.

Remark 4.1.5 We provide now rationales for the use of U and $\lhd$, and for condition (1), (2), and (3) in definition 4.1.3. First, the subset U of W contains all possible worlds that force some assertive formula η and thus $\lhd$ serves the purpose to reduce the forcing of a normative formula η° to the forcing of the underlying assertive formula η. Second, condition (1) of definition 4.1.3 makes sure that each $w \in W$ that forces ϵ also forces ϵ°. Similarly, condition (2) makes sure that each $w \in W$ that forces $\eta_1^\circ \circ \eta_2^\circ$ also forces $(\eta_1 \circ \eta_2)^\circ$. Hence, both conditions taken together guarantee that the $(-)^\circ$ operator preserves the relevant structure. Third, condition (3) of definition 4.1.3 will play a crucial role in the proof of soundness given below. It provides the semantic counterpart of the $(-)^\circ$ rule. It is worth mentioning that condition (1) together with condition (3) have the side effect that each $w \in W$ that forces η also forces η°, thus reflecting the fact that $\eta \Rightarrow_{\mathrm{d}} \eta^\circ$ is derivable in sequent calculus.

[1]Downward closure means that, for all $u, u' \in U$, if $u' \in \eta_p^\star$ and $u \preceq u'$ then $u \in \eta_p^\star$.

Lemma 4.1.6 For all $w, w' \in W$, if $w' \Vdash \gamma$ and $w \preceq w'$ then $w \Vdash \gamma$.

Proof By induction on the complexity of γ. We consider only the case where $\gamma = \eta^\circ$. Given $u' \in U$ such that $u' \lhd w'$, by condition (3) of definition 4.1.3 we have that, for all $u \in U$, if $u \lhd w$ then $u \preceq u'$. By condition (3) of definition 4.1.4 and the induction hypothesis we have that $u \Vdash \eta$ and thus, by condition (3) of definition 4.1.4 again, that $w \Vdash \eta^\circ$. $\square$

4.2 Soundness and completeness

We show now that the class of Kripke models given above is sound and complete (see theorems 4.2.3 and 4.2.4) with respect to the sequent calculus system provided in the previous chapter.

Remark 4.2.1 Given $\Gamma = \gamma_1, \ldots, \gamma_n$ we use $w \Vdash \Gamma$ as a shorthand notation for $w \Vdash \gamma_1 \circ \cdots \circ \gamma_n$.

Definition 4.2.2 A sequent $\Gamma \Rightarrow_{\mathtt{x}} \gamma$ is *valid* if, for all basic Kripke models $(W, U, \lhd, \Vdash)$ and for all $w \in W$ such that $w \Vdash \Gamma$, we have that $w \Vdash \gamma$.

Theorem 4.2.3 (soundness) If a sequent $\Gamma \Rightarrow_{\mathtt{x}} \gamma$ is derivable according to the rules of figure 3.1 then it is valid.

Proof This is shown by induction on the height of the derivation of $\Gamma \Rightarrow_{\mathtt{x}} \gamma$. The two most interesting cases are the ones in which the derivation ends with either an instance of con or an instance of $(-)^\circ$. In these cases we proceed as follows:

1. If the derivation ends with an instance of

$$\frac{\gamma', \gamma', \Gamma \Rightarrow_{\mathtt{x}} \gamma}{\gamma', \Gamma \Rightarrow_{\mathtt{x}} \gamma}\ \mathtt{con}$$

   then we have to show that if $w \Vdash \gamma', \Gamma$ then $w \Vdash \gamma$. By condition (5) of definition 4.1.4 there exist $w', w'' \in W$ such that $w \preceq w' \cdot w''$, $w' \Vdash \gamma'$, and $w'' \Vdash \Gamma$. By definition 4.1.3 we have that $(W, \cdot, 1, \preceq)$ is relevant and thus that $w' \preceq w' \cdot w'$. Hence, by condition (5) of definition 4.1.4 we have that $w' \Vdash \gamma' \circ \gamma'$. That $w \Vdash \gamma$ follows then from the induction hypothesis.

2. If the derivation ends with an instance of

$$\frac{\vec{\eta}, \Xi \Rightarrow_{\mathtt{c}} \eta}{(\vec{\eta})^\circ, \Xi \Rightarrow_{\mathtt{d}} \eta^\circ}\ (-)^\circ$$

then we have to show that if $w \Vdash (\vec{\eta})^\circ, \Xi$ then $w \Vdash \eta^\circ$. By condition (5) of definition 4.1.4 there exist $w', w'' \in W$ such that $w \preceq w' \cdot w''$, $w' \Vdash (\vec{\eta})^\circ$, and $w'' \Vdash \Xi$. Given $u' \in U$ such that $u' \lhd w'$, by applying first condition (5) of definition 4.1.4, then condition (2) of definition 4.1.3, then condition (3) of definition 4.1.4, and then condition (5) of definition 4.1.4 again, we get that $u' \Vdash \vec{\eta}$. Thus, by the induction hypothesis we have that $u' \cdot w'' \Vdash \eta$. That $w \Vdash \eta^\circ$ follows then by applying lemma 4.1.6 and condition (3) of definition 4.1.4 to the fact that, by condition (3) of definition 4.1.3, we have that, for all $u \in U$, if $u \lhd w$ then $u \preceq u' \cdot w''$. □

Theorem 4.2.4 (completeness) If a sequent $\Gamma \Rightarrow_x \gamma$ is valid then it is derivable according to the rules of figure 3.1.

Proof This is shown by constructing a so-called syntactic model. If $|\Xi|$ and $|\Gamma|$ denote the set of all pure causal and causal-deontic formulae, respectively, then clearly $(|\Xi|, \circ, \epsilon)$ and $(|\Gamma|, \circ, \epsilon)$ can be regarded as commutative monoids and the former is a proper submonoid of the latter. If the relation $\preceq \subseteq |\Gamma| \times |\Gamma|$ (resp. $|\Xi| \times |\Xi|$) is defined as

$$\{(\gamma_1, \gamma_2) \mid \gamma_1 \Rightarrow_x \gamma_2 \text{ is derivable without cuts}\}$$
$$(\text{resp. } \{(\xi_1, \xi_2) \mid \xi_1 \Rightarrow_x \xi_2 \text{ is derivable without cuts}\})$$

then $|\Xi| = (|\Xi|, \circ, \epsilon, \preceq)$ and $|\Gamma| = (|\Gamma|, \circ, \epsilon, \preceq)$ are both relevant monoids and the former is a proper relevant submonoid of the latter. Given $k, l \geq 1$, the relation $\lhd \subseteq |\Xi| \times |\Gamma|$ is obtained by taking the set of all ordered pairs of the form[2] $(\eta^{(k)}, \eta^\circ)$ and $(\xi^{(l)}, \xi)$ and closing it with respect to products, i.e. if $\xi_1 \lhd \gamma_1$ and $\xi_2 \lhd \gamma_2$, then $\xi_1 \circ \xi_2 \lhd \gamma_1 \circ \gamma_2$. Thus, we have a Kripke frame $(|\Gamma|, |\Xi|, \lhd)$. To obtain a Kripke model it suffices to provide the definition of the forcing relation $\Vdash \subseteq |\Gamma| \times |\Gamma|$: it is the set of all ordered pairs (γ_1, γ_2) such that $\gamma_1 \Rightarrow_x \gamma_2$ is derivable without cuts. If $\eta_p^\star$ denotes the set of all causal formulae ξ such that $\xi \Rightarrow_x \eta_p$ is derivable without cuts then it is straightforward to check that $\Vdash$ satisfies all the required conditions. For illustrative purposes let us spell out the proof of condition (3) of definition 4.1.4 in detail.

1. On the one hand, we have to show that if $w \Vdash \eta^\circ$ then, for all $u \in U$ such that $u \lhd w$, $u \Vdash \eta$. Assume that $w \Vdash \eta^\circ$, i.e. that $w = \gamma$ and that there exists a cut-free derivation of $\gamma \Rightarrow_d \eta^\circ$. By inspection of the rules we have that every cut-free derivation of $\gamma \Rightarrow_d \eta^\circ$ contains exactly one instance of the $(-)^\circ$ rule. So, if $\gamma = \eta_1^\circ \circ \cdots \circ \eta_n^\circ \circ \xi_1 \circ \cdots \circ \xi_m$

[2] We use $\gamma^{(n)}$ as shorthand notation for $\gamma \circ \cdots \circ \gamma$, the relevant conjunction of n times γ.

then every u such that $u \lhd w$ is of the form $u = \eta_1^{(k_1)} \circ \cdots \circ \eta_n^{(k_n)} \circ \xi_1^{(l_1)} \circ \cdots \circ \xi_m^{(l_m)}$ where $k_i, l_j \geq 1$. Now, if we assume furthermore that $u \not\Vdash \eta$ then there exists no cut-free derivation of $\eta_1^{(k_1)}, \ldots, \eta_n^{(k_n)}, \xi_1^{(l_1)}, \ldots, \xi_m^{(l_m)} \Rightarrow_\mathtt{c} \eta$. But the premise of the instance of $(-)^\circ$ has to be the conclusion of such a derivation, thus contradicting the last assumption. Therefore, it follows that $u \Vdash \eta$.

2. On the other hand, we have to show that if, for all $u \in U$ such that $u \lhd w$, $u \Vdash \eta$ then $w \Vdash \eta^\circ$. Given u such that $u \lhd w$ implies $u \Vdash \eta$, i.e. given u such that $u = \xi$ and $u \lhd w$ for some $w = \gamma$ imply that there exists a cut-free derivation of $\xi \Rightarrow_\mathtt{c} \eta$, we may assume that the derivation of $\xi \Rightarrow_\mathtt{c} \eta$ ends with $\vec{\eta}, \Xi \Rightarrow_\mathtt{c} \eta$ followed by instances of $\mathcal{L}\circ$ and $\mathtt{con}$. Within this context, the $(-)^\circ$ rule yields $(\vec{\eta})^\circ, \Xi \Rightarrow_\mathtt{d} \eta^\circ$ from which $\gamma \Rightarrow_\mathtt{d} \eta^\circ$ follows by instances of $\mathcal{L}\circ$ and $\mathtt{con}$. In other words, we have shown that, for all u such that $u = \xi$ and $u \lhd w$ for some $w = \gamma$, $w \Vdash \eta^\circ$. □

4.3 Discussion and final remarks

We have presented a sound and complete Kripke semantics for the basic causal-deontic fragment of the pragmatic language. We have not investigated whether and how the Kripke semantics considered in [BR03] is related to the one above and leave this issue for future research.

Chapter 5

Type-theoretical aspects

As a first step towards the development of a semantics of proofs we consider the basic causal-deontic system from a type-theoretical perspective, i.e. we introduce a presentation of the causal-deontic fragment of the pragmatic language that is based on terms and equations in context and discuss some of the issues related to this way of looking at the system. The advantage of having a type theory is that it provides us with the right level of generality for the investigation of a categorical semantics in the next chapter, i.e. terms in context can be regarded as morphisms and equations in context as equalities of morphisms. It is worth pointing out that the term formation rules are based on a natural deduction system that is derived from the sequent calculus presentation of the logic given in chapter 3. As an immediate consequence, the equations in context can be motivated by certain proof transformations given in chapter 3.

5.1 Terms and equations in context

Roughly speaking, the essence of the Curry-Howard correspondence consists of the insight that there is a tight connection between logical systems and specific extensions of the well known λ-calculus. In a type-theoretical setting this means that formulae, which in this context are also called types, can be annotated by terms that provide means to encode derivations as expressions of some term calculus. The aim of this section is to provide such a calculus (see definition 5.1.1) and to show how a natural deduction system for the types of the basic causal-deontic fragment of the pragmatic language can be annotated by its terms (see figure 5.1). However, this is only one aspect of the story. The other important issue is that one needs to consider also certain equations between expressions of the term calculus: we simply provide a collection of β- and η-equations (see figures 5.2 and 5.3), postponing their proof-theoretical justification to the next section. Finally, we note that one can dispense with equations induced by commuting conversions and that the term calculus is non-trivial (see remarks 5.1.7 and 5.1.10).

Definition 5.1.1 The term calculus for the basic causal deontic system is defined as follows:

$$\begin{aligned}\mathtt{M}, \mathtt{N} := \mathtt{u} \mid * \mid \lambda \mathtt{u}.\mathtt{M} \mid (\mathtt{M})\mathtt{N} \mid \mathtt{M}\cdot\mathtt{N} \mid \mathtt{let}\ \mathtt{N}\ \mathtt{be}\ *\ \mathtt{in}\ \mathtt{M} \mid \mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{u}_1\cdot\mathtt{u}_2\ \mathtt{in}\ \mathtt{M} \\ \mid \{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}] \mid \{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}] \mid \mathtt{prm\_ob}\,[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}]\end{aligned}$$

Remark 5.1.2 It is worth mentioning that the calculus can naturally be divided in two parts: whereas the terms of the first row of the previous definition correspond to the standard terms of the multiplicative fragment of intuitionistic linear logic (see for instance [MMdPR05]), the terms of the second row need some explanation. First, note that $\underline{\mathtt{M}}$ and $\underline{\mathtt{u}}$ are shorthand notation for $\mathtt{M}_1, \ldots, \mathtt{M}_n$ and $\mathtt{u}_1, \ldots, \mathtt{u}_n$. Then, let us consider the term formation rules or terms in contexts provided in figure 5.1 where contexts Γ, i.e. items on the lefthand side of $\rhd_{\mathtt{x}}$ in $\Gamma \rhd_{\mathtt{x}} \mathtt{M}{:}\,\gamma$, are collections $\mathtt{u}_1{:}\,\gamma_1, \ldots, \mathtt{u}_n{:}\,\gamma_n$ of typed variables with $\mathtt{u}_i \neq \mathtt{u}_j$ whenever $i \neq j$. The term formation rules for $\{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}]$, $\{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}]$, and $\mathtt{prm\_ob}\,[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}]$ are basically inspired by the term formation rule provided in [BdPR01] for a basic constructive modality; the square bracket notation is used to make explicit which assumptions are closed for substitution. Note that prm_ob stands for *promote obligation* and that its term formation rule is more complex than the one for the other two terms since it has to carry along parameters. Finally, it is worth pointing out that there is no special syntax for contractions in the term calculus. These are dealt with by allowing for sharing contexts in term formation rules with more than one premise. Indeed, we have that Γ'' stands for $\{\mathtt{u}{:}\,\gamma \mid (\mathtt{u}{:}\,\gamma) \in \Gamma''_i \text{ and } i \in \{1, \ldots, m+n\}\}$ in the $(-)^\circ$ rule; analogous conventions apply for Ξ' and Γ' in the $(\mathtt{a}/\mathtt{c})$ and $(\mathtt{c}/\mathtt{d})$ rules.

Remark 5.1.3 We have mentioned that the term formation rule for the term $\mathtt{prm\_ob}\,[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathtt{M}]$ depends on parameters. It seems very natural to make this more explicit by splitting contexts such as $\Gamma = \Gamma_1, \Gamma_2$ in two parts $\Gamma_1 \mid \Gamma_2$ where the context Γ_1 contains all parameters. The paradigmatic case concerns essentially the $(-)^\circ$ rule and requires to consider the following modified instance of it

$$\frac{\{\Gamma_i \mid - \rhd_{\mathtt{d}} \mathtt{M}_i{:}\,\xi_i\}_{i \in \mathrm{N}} \qquad \{\Gamma'_i \mid \Gamma''_i \rhd_{\mathtt{d}} \mathtt{M}'_i{:}\,\eta_i^\circ\}_{i \in \mathrm{M}} \qquad \underline{\mathtt{u}}{:}\,\Xi \mid \underline{\mathtt{u}}'{:}\,\vec{\eta} \rhd_{\mathtt{c}} \mathtt{M}{:}\,\eta}{\Gamma_1, \ldots, \Gamma_n, \Gamma'_1, \ldots, \Gamma'_m \mid \Gamma''_1, \ldots, \Gamma''_m \rhd_{\mathtt{d}} \mathtt{prm\_ob}\,[\underline{\mathtt{M}}, \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \underline{\mathtt{u}}' \mid \mathtt{M}]{:}\,\eta^\circ}$$

$$\frac{\Xi_1, \mathtt{u}{:}\,\eta, \Xi_2 \rhd_{\mathtt{c}} \mathtt{M}{:}\,\xi}{\Xi_1, \Xi_2 \rhd_{\mathtt{c}} \lambda \mathtt{u}.\mathtt{M}{:}\,\eta \Supset \xi} \Supset\mathcal{I} \qquad \frac{\Xi, \Xi_1 \rhd_{\mathtt{c}} \mathtt{M}_1{:}\,\eta \Supset \xi \quad \Xi, \Xi_2 \rhd_{\mathtt{c}} \mathtt{M}_2{:}\,\eta}{\Xi, \Xi_1, \Xi_2 \rhd_{\mathtt{c}} (\mathtt{M}_1)\mathtt{M}_2{:}\,\xi} \Supset\mathcal{E}$$

$$\frac{\Xi'_1, \Xi_1 \rhd_{\mathtt{c}} \mathtt{M}_1{:}\,\eta_1 \quad \cdots \quad \Xi'_n, \Xi_n \rhd_{\mathtt{c}} \mathtt{M}_n{:}\,\eta_n \quad \mathtt{u}_1{:}\,\eta_1, \ldots, \mathtt{u}_n{:}\,\eta_n \rhd_{\mathtt{a}} \mathtt{M}{:}\,\eta}{\Xi', \Xi_1, \ldots, \Xi_n \rhd_{\mathtt{c}} \{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}]{:}\,\eta} \,(\mathtt{a/c})$$

$$\frac{\Gamma'_1, \Gamma_1 \rhd_{\mathtt{d}} \mathtt{M}_1{:}\,\xi_1 \quad \cdots \quad \Gamma'_n, \Gamma_n \rhd_{\mathtt{d}} \mathtt{M}_n{:}\,\xi_n \quad \mathtt{u}_1{:}\,\xi_1, \ldots, \mathtt{u}_n{:}\,\xi_n \rhd_{\mathtt{c}} \mathtt{M}{:}\,\xi}{\Gamma', \Gamma_1, \ldots, \Gamma_n \rhd_{\mathtt{d}} \{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}]{:}\,\xi} \,(\mathtt{c/d})$$

$$\frac{\{\Gamma''_i, \Gamma_i \rhd_{\mathtt{d}} \mathtt{M}_i{:}\,\eta_i^\circ\}_{i \in \mathrm{N}} \quad \{\Gamma''_{i+n}, \Gamma'_i \rhd_{\mathtt{d}} \mathtt{M}'_i{:}\,\xi_i\}_{i \in \mathrm{M}} \quad \underline{\mathtt{u}}{:}\,\vec{\eta}, \underline{\mathtt{u}}'{:}\,\Xi \rhd_{\mathtt{c}} \mathtt{M}{:}\,\eta}{\Gamma'', \Gamma_1, \ldots, \Gamma_n, \Gamma'_1, \ldots, \Gamma'_m \rhd_{\mathtt{d}} \mathtt{prm\_ob}\,[\underline{\mathtt{M}}, \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \underline{\mathtt{u}}' \,|\, \mathtt{M}]{:}\,\eta^\circ} \,(-)^\circ$$

$$\frac{}{\mathtt{u}{:}\,\gamma \rhd_{\mathtt{x}} \mathtt{u}{:}\,\gamma}\,\mathtt{ax} \qquad \frac{}{\rhd_{\mathtt{x}} *{:}\,\epsilon}\,\epsilon\mathcal{I} \qquad \frac{\Gamma, \Gamma_1 \rhd_{\mathtt{x}} \mathtt{N}{:}\,\epsilon \quad \Gamma, \Gamma_2 \rhd_{\mathtt{x}} \mathtt{M}{:}\,\gamma}{\Gamma, \Gamma_1, \Gamma_2 \rhd_{\mathtt{x}} \mathtt{let\ N\ be\ *\ in\ M}{:}\,\gamma}\,\epsilon\mathcal{E}$$

$$\frac{\Gamma, \Gamma_1 \rhd_{\mathtt{x}} \mathtt{M}_1{:}\,\gamma_1 \quad \Gamma, \Gamma_2 \rhd_{\mathtt{x}} \mathtt{M}_2{:}\,\gamma_2}{\Gamma, \Gamma_1, \Gamma_2 \rhd_{\mathtt{x}} \mathtt{M}_1 \cdot \mathtt{M}_2{:}\,\gamma_1 \circ \gamma_2}\,\circ\mathcal{I} \qquad \frac{\Gamma, \Gamma' \rhd_{\mathtt{x}} \mathtt{N}{:}\,\gamma_1 \circ \gamma_2 \quad \Gamma, \Gamma_1, \mathtt{u}_1{:}\,\gamma_1, \mathtt{u}_2{:}\,\gamma_2, \Gamma_2 \rhd_{\mathtt{x}} \mathtt{M}{:}\,\gamma}{\Gamma, \Gamma_1, \Gamma', \Gamma_2 \rhd_{\mathtt{x}} \mathtt{let\ N\ be\ u_1 \cdot u_2\ in\ M}{:}\,\gamma}\,\circ\mathcal{E}$$

Figure 5.1: Formation rules for terms

where implicit contractions, i.e. sharing contexts, have been omitted for simplicity. Parameters occur on the lefthand side of the separator | and, in the case of the above rule, tell us which typed variables are not substituted by terms of type η_i°. It is an easy exercise to reformulate the term formation rules in such a way that the parameters are made explicit in the context.

Remark 5.1.4 We omit the formal definition of the substitution $\mathtt{M}[\mathtt{N}/\mathtt{u}]$ of a term $\mathtt{N}$ for the free variable $\mathtt{u}$ in the term $\mathtt{M}$ and only point out that the free variables $\underline{\mathtt{u}}$ in $\mathtt{M}$ become bound variables in $\{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}]$, $\{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}]$, and $\mathtt{prm\_ob}\,[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}]$. As an immediate consequence, substitution applies only to the terms in $\underline{\mathtt{M}}$ but not to the term $\mathtt{M}$.

Lemma 5.1.5 The substitution rule

$$\frac{\Gamma, \Gamma' \rhd_{\mathtt{x}} \mathtt{M}'{:}\,\gamma' \quad \Gamma, \Gamma_1, \mathtt{u}{:}\,\gamma', \Gamma_2 \rhd_{\mathtt{x}} \mathtt{M}{:}\,\gamma}{\Gamma, \Gamma_1, \Gamma', \Gamma_2 \rhd_{\mathtt{x}} \mathtt{M}[\mathtt{M}'/\mathtt{u}]{:}\,\gamma}\,\mathtt{sub}$$

is admissible in the system of figure 5.1.

Proof By induction on the height of the derivation of the right premise. $\square$

$$(\lambda \mathtt{u}.\mathtt{M})\mathtt{N} =_{\mathtt{c}} \mathtt{M}[\mathtt{N}/\mathtt{u}]$$

$$\mathtt{let} * \mathtt{be} * \mathtt{in}\ \mathtt{M} =_{\mathtt{x}} \mathtt{M}$$

$$\mathtt{let}\ \mathtt{M}_1 \cdot \mathtt{M}_2\ \mathtt{be}\ \mathtt{u}_1 \cdot \mathtt{u}_2\ \mathtt{in}\ \mathtt{M} =_{\mathtt{x}} \mathtt{M}[\mathtt{M}_1/\mathtt{u}_1, \mathtt{M}_2/\mathtt{u}_2]$$

$$\{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}_1, (\{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}']), \underline{\mathtt{M}}_2/\underline{\mathtt{u}}_1, \mathtt{u}, \underline{\mathtt{u}}_2 \,|\, \mathtt{M}] =_{\mathtt{c}}$$
$$\{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}_1, \underline{\mathtt{M}}, \underline{\mathtt{M}}_2/\underline{\mathtt{u}}_1, \underline{\mathtt{u}}, \underline{\mathtt{u}}_2 \,|\, \mathtt{M}[\mathtt{M}'/\mathtt{u}]]$$

$$\{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}_1, (\{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}']), \underline{\mathtt{M}}_2/\underline{\mathtt{u}}_1, \mathtt{u}, \underline{\mathtt{u}}_2 \,|\, \mathtt{M}] =_{\mathtt{d}}$$
$$\{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}_1, \underline{\mathtt{M}}, \underline{\mathtt{M}}_2/\underline{\mathtt{u}}_1, \underline{\mathtt{u}}, \underline{\mathtt{u}}_2 \,|\, \mathtt{M}[\mathtt{M}'/\mathtt{u}]]$$

$$\mathtt{prm\_ob}\,[\underline{\mathtt{M}}_1, (\{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \,|\, \mathtt{M}']), \underline{\mathtt{M}}_2/\underline{\mathtt{u}}_1, \mathtt{u}, \underline{\mathtt{u}}_2 \,|\, \mathtt{M}] =_{\mathtt{d}}$$
$$\mathtt{prm\_ob}\,[\underline{\mathtt{M}}_1, \underline{\mathtt{M}}, \underline{\mathtt{M}}_2/\underline{\mathtt{u}}_1, \underline{\mathtt{u}}, \underline{\mathtt{u}}_2 \,|\, \mathtt{M}[\mathtt{M}'/\mathtt{u}]]$$

$$\mathtt{prm\_ob}\,[\underline{\mathtt{M}}_2, (\mathtt{prm\_ob}\,[\underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1/\underline{\mathtt{u}}_1, \underline{\mathtt{u}}'_1 \,|\, \mathtt{M}']), \underline{\mathtt{M}}'_2/\underline{\mathtt{u}}_2, \mathtt{u}, \underline{\mathtt{u}}'_2 \,|\, \mathtt{M}] =_{\mathtt{d}}$$
$$\mathtt{prm\_ob}\,[\underline{\mathtt{M}}_2, \underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1, \underline{\mathtt{M}}'_2/\underline{\mathtt{u}}_2, \underline{\mathtt{u}}_1, \underline{\mathtt{u}}'_1, \underline{\mathtt{u}}'_2 \,|\, \mathtt{M}[\mathtt{M}'/\mathtt{u}]]$$

Figure 5.2: β-equations between terms

Remark 5.1.6 We turn now to consider the equalities that hold between certain expressions of the term calculus, i.e. we provide a collection of so-called equations in context $\Gamma \rhd_{\mathtt{x}} \mathtt{M} = \mathtt{N}\colon \gamma$. In figures 5.2 and 5.3 we limit ourselves to provide equations of the form $\mathtt{M} =_{\mathtt{x}} \mathtt{N}$ only, leaving it to the reader to supply the missing details (see also remarks 5.2.5 and 5.2.6 below). The first three equations of each figure are the standard β- and η-equalities of the multiplicative fragment of intuitionistic linear logic (see for instance [MMdPR05]). All other equations are similar to the ones provided by Kakutani in [Kak07] for an intuitionstic modal logic.

Remark 5.1.7 Readers familiar with linear logic will have noticed that we have omitted equations induced by so-called commuting conversions. As pointed out in [MMdPR05] these equations can be expressed in terms of the β- and η-equations for the relevant conjunction $\circ$ and its unit ϵ and thus we can dispense with them. To present the argument in a succinct form we introduce terms with a hole $C_{\mathtt{x}}[-]$, sometimes also called contexts (the details are provided in definition 5.1.8). As a consequence we have that all these equations can be regarded as instances of the general scheme

$$C_{\mathtt{x}}[\mathtt{let}\ \mathtt{N}\ \mathtt{be}\ p\ \mathtt{in}\ \mathtt{N}'] =_{\mathtt{x}} \mathtt{let}\ \mathtt{N}\ \mathtt{be}\ p\ \mathtt{in}\ C_{\mathtt{x}}[\mathtt{N}'] \tag{5.1}$$

$$\lambda \mathtt{u}.(\mathtt{M})\mathtt{u} =_{\mathtt{c}} \mathtt{M}$$

$$\mathtt{let\ N\ be} * \mathtt{in\ M}[*/\mathtt{u}] =_{\mathtt{x}} \mathtt{M}[\mathtt{N}/\mathtt{u}]$$

$$\mathtt{let\ N\ be\ u_1 \cdot u_2\ in\ M}[\mathtt{u_1 \cdot u_2/u}] =_{\mathtt{x}} \mathtt{M}[\mathtt{N}/\mathtt{u}]$$

$$\{\mathtt{a} \mapsto \mathtt{c}\}[\mathtt{M}/\mathtt{u} \mid \mathtt{u}] =_{\mathtt{c}} \mathtt{M}$$

$$\{\mathtt{c} \mapsto \mathtt{d}\}[\mathtt{M}/\mathtt{u} \mid \mathtt{u}] =_{\mathtt{d}} \mathtt{M}$$

$$\mathtt{prm\_ob}\,[\mathtt{M}/\mathtt{u} \mid \mathtt{u}] =_{\mathtt{d}} \mathtt{M}$$

Figure 5.3: η-equations between terms

where: (a) $\mathcal{C}_{\mathtt{x}}[\mathtt{M}]$ denotes the term obtained by replacing the hole $[-]$ in $\mathcal{C}_{\mathtt{x}}[-]$ with the term $\mathtt{M}$; (b) p stands either for $\mathtt{v_1 \cdot v_2}$ or $*$, depending on whether we deal with relevant conjunction or its unit. The calculations are provided in the proof of lemma 5.1.9.

Definition 5.1.8 Terms with a hole $\mathcal{C}_{\mathtt{x}}[-]$ are defined by the following collection of grammars.

$$\mathcal{C}_{\mathtt{a}}[-] := [-] \mid \mathcal{C}_{\mathtt{a}}[-]\cdot\mathtt{M} \mid \mathtt{M}\cdot\mathcal{C}_{\mathtt{a}}[-] \mid \mathtt{let}\ \mathcal{C}_{\mathtt{a}}[-]\ \mathtt{be}\ p\ \mathtt{in\ M} \mid \mathtt{let\ M\ be}\ p\ \mathtt{in}\ \mathcal{C}_{\mathtt{a}}[-]$$

$$\begin{aligned}\mathcal{C}_{\mathtt{c}}[-] := {} & [-] \mid \mathcal{C}_{\mathtt{c}}[-]\cdot\mathtt{M} \mid \mathtt{M}\cdot\mathcal{C}_{\mathtt{c}}[-] \mid \mathtt{let}\ \mathcal{C}_{\mathtt{c}}[-]\ \mathtt{be}\ p\ \mathtt{in\ M} \mid \mathtt{let\ M\ be}\ p\ \mathtt{in}\ \mathcal{C}_{\mathtt{c}}[-] \\ & \mid \lambda\mathtt{u}.\mathcal{C}_{\mathtt{c}}[-] \mid (\mathcal{C}_{\mathtt{c}}[-])\,\mathtt{M} \mid (\mathtt{M})(\mathcal{C}_{\mathtt{c}}[-]) \mid \{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}, \mathcal{C}_{\mathtt{c}}[-], \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}' \mid \mathtt{M}] \\ & \mid \{\mathtt{a} \mapsto \mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathcal{C}_{\mathtt{c}}[-]]\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{\mathtt{d}}[-] := {} & [-] \mid \mathcal{C}_{\mathtt{d}}[-]\cdot\mathtt{M} \mid \mathtt{M}\cdot\mathcal{C}_{\mathtt{d}}[-] \mid \mathtt{let}\ \mathcal{C}_{\mathtt{d}}[-]\ \mathtt{be}\ p\ \mathtt{in\ M} \mid \mathtt{let\ M\ be}\ p\ \mathtt{in}\ \mathcal{C}_{\mathtt{d}}[-] \\ & \mid \{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}, \mathcal{C}_{\mathtt{d}}[-], \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}' \mid \mathtt{M}] \mid \mathtt{prm\_ob}\,[\underline{\mathtt{M}}, \mathcal{C}_{\mathtt{d}}[-], \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}' \mid \mathtt{M}] \\ & \mid \{\mathtt{c} \mapsto \mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathcal{C}_{\mathtt{d}}[-]] \mid \mathtt{prm\_ob}\,[\underline{\mathtt{M}}/\underline{\mathtt{u}} \mid \mathcal{C}_{\mathtt{d}}[-]]\end{aligned}$$

Lemma 5.1.9 Instances of (5.1) are induced by the equations provided in figures 5.2 and 5.3.

Proof We only treat the case where $p = \mathtt{v_1 \cdot v_2}$, the other one being analogous.

$$\begin{aligned}\mathtt{let\ N\ be\ v_1\cdot v_2\ in}\ \mathcal{C}_{\mathtt{x}}[\mathtt{N}'] & =_{\mathtt{x}} \mathtt{let\ N\ be\ v_1\cdot v_2\ in}\ \mathcal{C}_{\mathtt{x}}[\mathtt{N}'[\mathtt{v_1/v_1, v_2/v_2}]] && (s) \\ & =_{\mathtt{x}} \mathtt{let\ N\ be\ v_1\cdot v_2\ in}\ \mathcal{C}_{\mathtt{x}}[\mathtt{let\ v_1\cdot v_2\ be\ v_1\cdot v_2\ in\ N}'] && (\beta) \\ & =_{\mathtt{x}} \mathtt{let\ N\ be\ v_1\cdot v_2\ in}\ (\mathcal{C}_{\mathtt{x}}[\mathtt{let\ v\ be\ v_1\cdot v_2\ in\ N}'][\mathtt{v_1\cdot v_2/v}]) && (s) \\ & =_{\mathtt{x}} \mathcal{C}_{\mathtt{x}}[\mathtt{let\ v\ be\ v_1\cdot v_2\ in\ N}'][\mathtt{N/v}] && (\eta) \\ & =_{\mathtt{x}} \mathcal{C}_{\mathtt{x}}[\mathtt{let\ N\ be\ v_1\cdot v_2\ in\ N}'] && (s)\end{aligned}$$

$$
\begin{aligned}
\mathtt{u}^{\ddagger} &= \mathtt{u}\\
*^{\ddagger} &= \langle\rangle\\
(\lambda \mathtt{u}.\mathtt{M})^{\ddagger} &= \lambda \mathtt{u}.\mathtt{M}^{\ddagger}\\
((\mathtt{M}_1)\mathtt{M}_2)^{\ddagger} &= (\mathtt{M}_1^{\ddagger})\mathtt{M}_2^{\ddagger}\\
(\mathtt{M}_1\cdot\mathtt{M}_2)^{\ddagger} &= \langle \mathtt{M}_1^{\ddagger}, \mathtt{M}_2^{\ddagger}\rangle\\
(\mathtt{let}\ \mathtt{N}\ \mathtt{be}\ *\ \mathtt{in}\ \mathtt{M})^{\ddagger} &= \mathtt{M}^{\ddagger}\\
(\mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{u}_1\cdot\mathtt{u}_2\ \mathtt{in}\ \mathtt{M})^{\ddagger} &= \mathtt{M}^{\ddagger}[\mathtt{fst}\ \mathtt{N}^{\ddagger}/\mathtt{u}_1, \mathtt{snd}\ \mathtt{N}^{\ddagger}/\mathtt{u}_2]\\
(\{\mathtt{a}\mapsto\mathtt{c}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}}\,|\,\mathtt{M}])^{\ddagger} &= \lambda k.(\underline{\mathtt{M}}^{\ddagger})(\lambda\underline{\mathtt{u}}.(k)\mathtt{M}^{\ddagger})\\
(\{\mathtt{c}\mapsto\mathtt{d}\}[\underline{\mathtt{M}}/\underline{\mathtt{u}}\,|\,\mathtt{M}])^{\ddagger} &= \lambda k.(\underline{\mathtt{M}}^{\ddagger})(\lambda\underline{\mathtt{u}}.(k)\mathtt{M}^{\ddagger})\\
(\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \underline{\mathtt{u}}'\,|\,\mathtt{M}])^{\ddagger} &= \lambda k.(\underline{\mathtt{M}}^{\ddagger})(\lambda\underline{\mathtt{u}}.((\underline{\mathtt{M}}')^{\ddagger})(\lambda\underline{\mathtt{u}}'.(k)\mathtt{M}^{\ddagger}))
\end{aligned}
$$

Figure 5.4: Translation into λ-calculus

For instance, given the case that $C_{\mathtt{d}}[-]$ is shorthand for $\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, [-], \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}]$, we get that

$$
\begin{aligned}
&\mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ (\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, \mathtt{N}', \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}])\\
&\quad =_{\mathtt{d}} \mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ (\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, \mathtt{N}'[\mathtt{v}_1/\mathtt{v}_1, \mathtt{v}_2/\mathtt{v}_2], \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}])\\
&\quad =_{\mathtt{d}} \mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ (\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, (\mathtt{let}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ \mathtt{N}'), \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}])\\
&\quad =_{\mathtt{d}} \mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ ((\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, (\mathtt{let}\ \mathtt{v}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ \mathtt{N}'), \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}])[\mathtt{v}_1\cdot\mathtt{v}_2/\mathtt{v}])\\
&\quad =_{\mathtt{d}} (\mathtt{prm\_ob}\,[\underline{\mathtt{M}}, (\mathtt{let}\ \mathtt{v}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ \mathtt{N}'), \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}])[\mathtt{N}/\mathtt{v}]\\
&\quad =_{\mathtt{d}} \mathtt{prm\_ob}\,[\underline{\mathtt{M}}, (\mathtt{let}\ \mathtt{N}\ \mathtt{be}\ \mathtt{v}_1\cdot\mathtt{v}_2\ \mathtt{in}\ \mathtt{N}'), \underline{\mathtt{M}}'/\underline{\mathtt{u}}, \mathtt{u}, \underline{\mathtt{u}}'\,|\,\mathtt{M}]
\end{aligned}
$$

□

Remark 5.1.10 We conclude this section with an equality-preserving translation into simple typed λ-calculus with pairing. This serves the purpose to show that the term assignment is non-trivial, i.e. that not all terms are equal. We simply adapt the translation given in [Bie94] and combine it with the one provided by Kakutani in [Kak07]: the translation of terms is given in figure 5.4. The translation of types or formulae needs to take care also of the deductive context, generally indicated by the entailment relation $\rhd_{\mathtt{x}}$ of our calculus. Given that $(\eta)^{\ddagger}_{\mathtt{c}} = ((\eta)^{\ddagger}_{\mathtt{a}} \supset \eta_p) \supset \eta_p$, $(\xi)^{\ddagger}_{\mathtt{d}} = ((\xi)^{\ddagger}_{\mathtt{c}} \supset \eta_p) \supset \eta_p$, and $(\eta^{\circ})^{\ddagger}_{\mathtt{d}} = ((\eta)^{\ddagger}_{\mathtt{c}} \supset \eta_p) \supset \eta_p$ where η_p is an arbitrary assertive elementary formula, it is straightforward to show that if $\Gamma \rhd_{\mathtt{x}} \mathtt{M}\colon \gamma$ then $(\Gamma)^\ddagger_{\mathtt{x}} \rhd \mathtt{M}^\ddagger\colon (\gamma)^\ddagger_{\mathtt{x}}$ is derivable in simple typed λ-calculus with pairing, i.e. that the translation preserves typing. That the translation also preserves equalities is shown by the following result.

Lemma 5.1.11 If $\Gamma \rhd_{\mathtt{x}} \mathtt{M} = \mathtt{N}\colon \gamma$ then $(\Gamma)^\ddagger_{\mathtt{x}} \rhd \mathtt{M}^\ddagger = \mathtt{N}^\ddagger\colon (\gamma)^\ddagger_{\mathtt{x}}$.

Proof We only provide the following calculation regarding β-equality:

$$\begin{aligned}
&(\mathtt{prm_ob}\,[\underline{\mathtt{M}}_2, (\mathtt{prm_ob}\,[\underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1/\underline{\mathtt{u}}_1, \underline{\mathtt{u}}'_1 \mid \mathtt{M}']), \underline{\mathtt{M}}'_2/\underline{\mathtt{u}}_2, \mathtt{u}, \underline{\mathtt{u}}'_2 \mid \mathtt{M}])^\ddagger \\
&\quad =_{def} \lambda k.(\underline{\mathtt{M}}^\ddagger_2)(\lambda \underline{\mathtt{u}}_2.(\underline{(\mathtt{prm_ob}\,[\underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1/\underline{\mathtt{u}}_1, \underline{\mathtt{u}}'_1 \mid \mathtt{M}'])^\ddagger})(\lambda \mathtt{u}.((\underline{\mathtt{M}}'_2)^\ddagger)(\lambda \underline{\mathtt{u}}'_2.(k)\mathtt{M}^\ddagger))) \\
&\quad =_{def} \lambda k.(\underline{\mathtt{M}}^\ddagger_2)(\lambda \underline{\mathtt{u}}_2.\underline{(\lambda h.(\underline{\mathtt{M}}^\ddagger_1)(\lambda \underline{\mathtt{u}}_1.((\underline{\mathtt{M}}'_1)^\ddagger)(\lambda \underline{\mathtt{u}}'_1.(h)(\mathtt{M}')^\ddagger)))(\lambda \mathtt{u}.((\underline{\mathtt{M}}'_2)^\ddagger)(\lambda \underline{\mathtt{u}}'_2.(k)\mathtt{M}^\ddagger)))} \\
&\quad =_{\beta} \lambda k.(\underline{\mathtt{M}}^\ddagger_2)(\lambda \underline{\mathtt{u}}_2.(\underline{\mathtt{M}}^\ddagger_1)(\lambda \underline{\mathtt{u}}_1.((\underline{\mathtt{M}}'_1)^\ddagger)(\lambda \underline{\mathtt{u}}'_1.\underline{((\lambda \mathtt{u}.((\underline{\mathtt{M}}'_2)^\ddagger)(\lambda \underline{\mathtt{u}}'_2.(k)\mathtt{M}^\ddagger)))(\mathtt{M}')^\ddagger}))) \\
&\quad =_{\beta} \lambda k.(\underline{\mathtt{M}}^\ddagger_2)(\lambda \underline{\mathtt{u}}_2.(\underline{\mathtt{M}}^\ddagger_1)(\lambda \underline{\mathtt{u}}_1.((\underline{\mathtt{M}}'_1)^\ddagger)(\lambda \underline{\mathtt{u}}'_1.((\underline{\mathtt{M}}'_2)^\ddagger)(\lambda \underline{\mathtt{u}}'_2.(k)(\mathtt{M}^\ddagger[(\mathtt{M}')^\ddagger/\mathtt{u}]))))) \\
&\quad =_{def} (\mathtt{prm_ob}\,[\underline{\mathtt{M}}_2, \underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1, \underline{\mathtt{M}}'_2/\underline{\mathtt{u}}_2, \underline{\mathtt{u}}_1, \underline{\mathtt{u}}'_1, \underline{\mathtt{u}}'_2 \mid \mathtt{M}[\mathtt{M}'/\mathtt{u}]])^\ddagger
\end{aligned}$$

□

5.2 Proof-theoretical considerations

The aim of this section is to provide a proof-theoretical justification of the type theory presented in the previous section. First of all, we need to address the issue of switching from the sequent calculus presentation of the logic given in chapter 3 to a natural deduction system: we show that whatever can be derived in one system can also be derived in the other system (see proposition 5.2.4), thus proving that the two presentations of the logic are equivalent. Next, we show that the equations given in figures 5.2 and 5.3 are induced by so-called β-reductions and η-expansions (see remarks 5.2.5 and 5.2.6). Finally, we also provide a proof-theoretical justification of scheme (5.1) considered in remark 5.1.7 (see remark 5.2.8).

Remark 5.2.1 Since the emphasis of this section is to provide a motivation for the type theory we limit ourselves to give only some hints with respect to the issues of normalization and confluence. A proof of strong normalization for β- and η-reductions can be obtained in a straightforward fashion by means of the translation provided in figure 5.4. Given that strong normalization holds, confluence follows from weak confluence via Newman's lemma (see for instance [TS00]): a proof of weak confluence for a system very similar to ours is given in [Kak07].

Remark 5.2.2 The natural deduction rules for the causal-deontic system are implicitly given in the term formation rules of figure 5.1 and can be obtained by keeping the types while erasing the variables and terms on the lefthand side and the righthand side of $\rhd_{\mathtt{x}}$. To make this more explicit we shall change notation and write $\gamma_1, \ldots, \gamma_n \vdash_{\mathtt{x}} \gamma$ instead of $\mathrm{u}_1 : \gamma_1, \ldots, \mathrm{u}_n : \gamma_n \rhd_{\mathtt{x}} \mathtt{M} : \gamma$ whenever we do not care about variables and terms.

Remark 5.2.3 The only non-standard rules of figure 5.1 are the $(\mathtt{a}/\mathtt{c})$, $(\mathtt{c}/\mathtt{d})$, and $(-)^\circ$ rules and they can all be motivated in a similar fashion. These rules are inspired by the natural deduction rule for the constructive modal logic **IK** given in [BdPR01] (see also [Bel85]) and have the advantage that the system is closed under substitution (see lemma 5.1.5). Indeed, consider the $(-)^\circ$ rule

$$\frac{\{\Gamma_i, \Gamma_i'' \vdash_{\mathtt{d}} \eta_i^\circ\}_{i \in \mathrm{N}} \quad \{\Gamma_i', \Gamma_{i+n}'' \vdash_{\mathtt{d}} \xi_i\}_{i \in \mathrm{M}} \quad \eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_m \vdash_{\mathtt{c}} \eta}{\Gamma_1, \ldots, \Gamma_n, \Gamma_1', \ldots, \Gamma_m', \Gamma'' \vdash_{\mathtt{d}} \eta^\circ}$$

written in tree form:

$$\frac{\eta_1^\circ \cdots \eta_n^\circ \quad \xi_1 \cdots \xi_m \qquad \begin{array}{c} [\![\eta_1]\!] \cdots [\![\eta_n]\!] \; [\![\xi_1]\!] \cdots [\![\xi_m]\!] \\ \vdots \\ \eta \end{array}}{\eta^\circ}$$

The $[\![-]\!]$ notation in the scheme indicates that all premises of the derivation ending with η must be closed for substitution, thus making it necessary to add the new premises $\eta_1^\circ, \ldots, \eta_n^\circ$ and $\xi_1, \ldots, \xi_m$ to the rule. In this way one can make sure that substitutions cannot inappropriately interfere with the different levels of the system. Note that the rule acts both as an introduction as well as an elimination rule: on the one hand it introduces η°; on the other hand it eliminates $\eta_1^\circ, \ldots, \eta_n^\circ$ and $\xi_1, \ldots, \xi_m$. The elimination of the pure causal formulae $\xi_1, \ldots, \xi_n$ is justified by the fact that there is an implicit change of entailment relation.

Proposition 5.2.4 (equivalence of systems) $\Gamma \vdash_{\mathtt{x}} \gamma$ if and only if $\Gamma \Rightarrow_{\mathtt{x}} \gamma$.

Proof This can be shown by providing a translation $\mathcal{N}(-)$ from sequent calculus derivations to natural deduction ones and a translation $\mathcal{G}(-)$ from natural deduction derivations to sequent calculus ones. Here we consider only the cases for the $(-)^\circ$ rule.

1. A sequent calculus derivation

$$\frac{\begin{array}{c}\pi \\ \eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_m \Rightarrow_{\mathtt{c}} \eta\end{array}}{\eta_1^\circ, \ldots, \eta_n^\circ, \xi_1, \ldots, \xi_m \Rightarrow_{\mathtt{d}} \eta^\circ}\ (-)^\circ$$

translates to

$$\dfrac{\left\{\dfrac{}{\eta_i^\circ \vdash_{\mathrm{d}} \eta_i^\circ}\right\}_{i\in \mathrm{N}} \quad \left\{\dfrac{}{\xi_i \vdash_{\mathrm{d}} \xi_i}\right\}_{i\in \mathrm{M}} \quad \begin{array}{c}\mathcal{N}(\pi)\\ \eta_1,\ldots,\eta_n,\xi_1,\ldots,\xi_m \vdash_{\mathrm{c}} \eta\end{array}}{\eta_1^\circ,\ldots,\eta_n^\circ,\xi_1,\ldots,\xi_m \vdash_{\mathrm{d}} \eta^\circ}\,(-)^\circ$$

2. A natural deduction derivation

$$\dfrac{\left\{\begin{array}{c}\pi_i\\ \Gamma_i'',\Gamma_i \vdash_{\mathrm{d}} \eta_i^\circ\end{array}\right\}_{i\in \mathrm{N}} \quad \left\{\begin{array}{c}\pi_i'\\ \Gamma_{i+n}'',\Gamma_i' \vdash_{\mathrm{d}} \xi_i\end{array}\right\}_{i\in \mathrm{M}} \quad \begin{array}{c}\pi\\ \eta_1,\ldots,\eta_n,\xi_1,\ldots,\xi_m \vdash_{\mathrm{c}} \eta\end{array}}{\Gamma'',\Gamma_1,\ldots,\Gamma_n,\Gamma_1',\ldots,\Gamma_m' \vdash_{\mathrm{d}} \eta^\circ}\,(-)^\circ$$

translates to

$$\dfrac{\dfrac{\left\{\begin{array}{c}\mathcal{G}(\pi_i)\\ \Gamma_i'',\Gamma_i \Rightarrow_{\mathrm{d}} \eta_i^\circ\end{array}\right\}_{i\in \mathrm{N}} \quad \left\{\begin{array}{c}\mathcal{G}(\pi_i')\\ \Gamma_{i+n}'',\Gamma_i \Rightarrow_{\mathrm{d}} \xi_i\end{array}\right\}_{i\in \mathrm{M}} \quad \dfrac{\begin{array}{c}\mathcal{G}(\pi)\\ \eta_1,\ldots,\eta_n,\xi_1,\ldots,\xi_m \Rightarrow_{\mathrm{c}} \eta\end{array}}{\eta_1^\circ,\ldots,\eta_n^\circ,\xi_1,\ldots,\xi_m \Rightarrow_{\mathrm{d}} \eta^\circ}\,(-)^\circ}{\Gamma_1'',\Gamma_1,\ldots,\Gamma_n'',\Gamma_n,\Gamma_{1+n}'',\Gamma_1',\ldots,\Gamma_{m+n}'',\Gamma_m' \Rightarrow_{\mathrm{d}} \eta^\circ}\,\mathrm{cut}}{\Gamma'',\Gamma_1,\ldots,\Gamma_n,\Gamma_1',\ldots,\Gamma_m' \Rightarrow_{\mathrm{d}} \eta^\circ}\,\mathrm{con}$$

□

Remark 5.2.5 The equations of figure 5.2 are induced by β-reductions. These arise when a certain formula is first introduced and then immediately eliminated at the next step, thus giving rise to a so-called detour in the derivation. In the case of the (linear) conjunction $\circ$ we have that the derivation

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma_1 \vdash_{\mathrm{x}} \gamma_1\end{array} \quad \begin{array}{c}\pi_2\\ \Gamma_2 \vdash_{\mathrm{x}} \gamma_2\end{array}}{\Gamma_1,\Gamma_2 \vdash_{\mathrm{x}} \gamma_1 \circ \gamma_2}\,\circ\mathcal{I} \quad \begin{array}{c}\pi\\ \Gamma,\gamma_1,\gamma_2,\Gamma' \vdash_{\mathrm{x}} \gamma\end{array}}{\Gamma,\Gamma_1,\Gamma_2,\Gamma' \vdash_{\mathrm{x}} \gamma}\,\circ\mathcal{E}$$

β-reduces to

$$\dfrac{\begin{array}{c}\pi_1\\ \Gamma_1 \vdash_{\mathrm{x}} \gamma_1\end{array} \quad \dfrac{\begin{array}{c}\pi_2\\ \Gamma_2 \vdash_{\mathrm{x}} \gamma_2\end{array} \quad \begin{array}{c}\pi\\ \Gamma,\gamma_1,\gamma_2,\Gamma' \vdash_{\mathrm{x}} \gamma\end{array}}{\Gamma,\gamma_1,\Gamma_2,\Gamma' \vdash_{\mathrm{x}} \gamma}\,\mathrm{sub}}{\Gamma,\Gamma_1,\Gamma_2,\Gamma' \vdash_{\mathrm{x}} \gamma}\,\mathrm{sub}$$

Since the (a/c), (c/d), and $(-)^\circ$ rules act both as an introduction as well as an elimination rule the concept of β-reductions applies to them as well. An example of a β-reduction involving two instances of the $(-)^\circ$ rule is given in figure 5.5 where implicit contractions, i.e. sharing contexts, have been omitted for simplicity.

$$\dfrac{\left\{\begin{matrix}\pi_{2i}\\ \Gamma_{2i}\vdash_{\mathrm{d}}\eta_{2i}^{\circ}\end{matrix}\right\}_{i\in\mathrm{M}} \qquad \dfrac{\left\{\begin{matrix}\pi_{1i}\\ \Gamma_{1i}\vdash_{\mathrm{d}}\eta_{1i}^{\circ}\end{matrix}\right\}_{i\in\mathrm{N}} \quad \left\{\begin{matrix}\pi'_{1i}\\ \Gamma'_{1i}\vdash_{\mathrm{d}}\xi_{1i}\end{matrix}\right\}_{i\in\mathrm{N}'} \quad \begin{matrix}\pi'\\ \vec{\eta}_1,\Xi_1\vdash_{\mathrm{c}}\eta'\end{matrix}}{\Gamma_{11}\ldots\Gamma_{1n},\Gamma'_{11},\ldots,\Gamma'_{1n'}\vdash_{\mathrm{d}}(\eta')^{\circ}}\,(-)^{\circ} \qquad \left\{\begin{matrix}\pi'_{2i}\\ \Gamma'_{2i}\vdash_{\mathrm{d}}\xi_{2i}\end{matrix}\right\}_{i\in\mathrm{M}'} \quad \begin{matrix}\pi\\ \vec{\eta}_2,\eta',\Xi_2\vdash_{\mathrm{c}}\eta\end{matrix}}{\Gamma_{21},\ldots,\Gamma_{2m},\Gamma_{11},\ldots\Gamma_{1n},\Gamma'_{11},\ldots\Gamma'_{1n'},\Gamma'_{21},\ldots,\Gamma'_{2m'}\vdash_{\mathrm{d}}\eta^{\circ}}\,(-)^{\circ}$$

β-reduces to

$$\dfrac{\left\{\begin{matrix}\pi_{2i}\\ \Gamma_{2i}\vdash_{\mathrm{d}}\eta_{2i}^{\circ}\end{matrix}\right\}_{j\in\mathrm{M}} \quad \left\{\begin{matrix}\pi_{1i}\\ \Gamma_{1i}\vdash_{\mathrm{d}}\eta_{1i}^{\circ}\end{matrix}\right\}_{i\in\mathrm{N}} \quad \left\{\begin{matrix}\pi'_{1i}\\ \Gamma'_{1i}\vdash_{\mathrm{d}}\xi_{1i}\end{matrix}\right\}_{i\in\mathrm{N}'} \quad \left\{\begin{matrix}\pi'_{2i}\\ \Gamma'_{2i}\vdash_{\mathrm{d}}\xi_{2i}\end{matrix}\right\}_{i\in\mathrm{M}'} \qquad \dfrac{\begin{matrix}\pi'\\ \vec{\eta}_1,\Xi_1\vdash_{\mathrm{c}}\eta'\end{matrix} \qquad \begin{matrix}\pi\\ \vec{\eta}_2,\eta',\Xi_2\vdash_{\mathrm{c}}\eta\end{matrix}}{\vec{\eta}_2,\vec{\eta}_1,\Xi_1,\Xi_2\vdash_{\mathrm{c}}\eta}\,\mathrm{sub}}{\Gamma_{21},\ldots,\Gamma_{2m},\Gamma_{11},\ldots\Gamma_{1n},\Gamma'_{11},\ldots\Gamma'_{1n'},\Gamma'_{21},\ldots,\Gamma'_{2m'}\vdash_{\mathrm{d}}\eta^{\circ}}\,(-)^{\circ}$$

Figure 5.5: Example of a β-reduction

Remark 5.2.6 The equations of figure 5.3 are induced by η-expansions. These arise when a certain formula is decomposed into simpler formulae. In the case of the (linear) conjunction $\circ$ we have that

$$\cfrac{\overset{\pi}{\Gamma \vdash_{x} \gamma_1 \circ \gamma_2} \qquad \overset{\pi'}{\Gamma_1, \gamma_1 \circ \gamma_2, \Gamma_2 \vdash_{x} \gamma}}{\Gamma_1, \Gamma, \Gamma_2 \vdash_{x} \gamma}\ \mathsf{sub}$$

η-expands to

$$\cfrac{\overset{\pi}{\Gamma \vdash_{x} \gamma_1 \circ \gamma_2} \qquad \cfrac{\cfrac{\cfrac{}{\gamma_1 \vdash_{x} \gamma_1} \quad \cfrac{}{\gamma_2 \vdash_{x} \gamma_2}}{\gamma_1, \gamma_2 \vdash_{x} \gamma_1 \circ \gamma_2}\ \circ\mathcal{I} \qquad \overset{\pi'}{\Gamma_1, \gamma_1 \circ \gamma_2, \Gamma_2 \vdash_{x} \gamma}}{\Gamma_1, \gamma_1, \gamma_2, \Gamma_2 \vdash_{x} \gamma}\ \mathsf{sub}}{\Gamma_1, \Gamma, \Gamma_2 \vdash_{x} \gamma}\ \circ\mathcal{E}$$

Note that we have build in a substitution into the η-expansion, thus getting a generalised version of it. If π' consists only of an axiom then the η-expansion for the (linear) conjunction $\circ$ gets the more standard form:

$$\overset{\pi}{\Gamma \vdash_{x} \gamma_1 \circ \gamma_2} \qquad \eta\text{-expands to} \qquad \cfrac{\overset{\pi}{\Gamma \vdash_{x} \gamma_1 \circ \gamma_2} \qquad \cfrac{\cfrac{}{\gamma_1 \vdash_{x} \gamma_1} \quad \cfrac{}{\gamma_2 \vdash_{x} \gamma_2}}{\gamma_1, \gamma_2 \vdash_{x} \gamma_1 \circ \gamma_2}\ \circ\mathcal{I}}{\Gamma \vdash_{x} \gamma_1 \circ \gamma_2}\ \circ\mathcal{E}$$

In the case of the $(-)^\circ$ rule we have that

$$\overset{\pi}{\Gamma \vdash_{d} \eta^\circ} \qquad \eta\text{-expands to} \qquad \cfrac{\overset{\pi}{\Gamma \vdash_{d} \eta^\circ} \qquad \cfrac{}{\eta \vdash_{c} \eta}}{\Gamma \vdash_{d} \eta^\circ}\ (-)^\circ$$

Analogously, we also have η-expansions for the (a/c) and (c/d) rules.

Remark 5.2.7 It is worth mentioning that there is a tight connection between β-equations and cut elimination: the interested reader may wish to verify that the left- and righthand sides of the last four equations of figure 5.2 correspond to the terms annotating the conclusions of the translations of the sequent calculus derivations given in the last case of the proof of theorem 3.1.8. Similarly, the last three η-equations of figure 5.3 are linked to the translations of the sequent calculus derivations given in the proof of lemma 3.1.7.

Remark 5.2.8 The equation in scheme (5.1) is induced by commuting conversions. Generally, these arise in the presence of sum-like elimination rules (see for instance [GLT90]). Roughly,

$$\dfrac{\left\{\begin{matrix}\pi_i\\ \Gamma_i \vdash_{\mathrm{d}} \eta_i^\circ\end{matrix}\right\}_{i\in \mathrm{N}} \qquad \dfrac{\begin{matrix}\pi'\\ \Gamma' \vdash_{\mathrm{d}} \gamma_1\circ\gamma_2\end{matrix} \qquad \begin{matrix}\pi''\\ \Gamma,\gamma_1,\gamma_2,\Gamma''\vdash_{\mathrm{d}} (\eta')^\circ\end{matrix}}{\Gamma,\Gamma',\Gamma''\vdash_{\mathrm{d}}(\eta')^\circ}\circ\varepsilon \qquad \left\{\begin{matrix}\pi_i'\\ \Gamma_i' \vdash_{\mathrm{d}} \xi_i\end{matrix}\right\}_{i\in \mathrm{N}'} \qquad \begin{matrix}\pi\\ \eta_1,\dots,\eta_n,\eta',\xi_1,\dots,\xi_{n'}\vdash_{\mathrm{c}}\eta\end{matrix}}{\Gamma_1,\dots,\Gamma_n,\Gamma,\Gamma',\Gamma'',\Gamma'_1,\dots,\Gamma'_{n'}\vdash_{\mathrm{d}}\eta^\circ}(-)^\circ$$

converts to

$$\dfrac{\begin{matrix}\pi'\\ \Gamma' \vdash_{\mathrm{d}} \gamma_1\circ\gamma_2\end{matrix} \qquad \dfrac{\left\{\begin{matrix}\pi_i\\ \Gamma_i \vdash_{\mathrm{d}} \eta_i^\circ\end{matrix}\right\}_{i\in \mathrm{N}} \qquad \begin{matrix}\pi''\\ \Gamma,\gamma_1,\gamma_2,\Gamma''\vdash_{\mathrm{d}} (\eta')^\circ\end{matrix} \qquad \left\{\begin{matrix}\pi_i'\\ \Gamma_i' \vdash_{\mathrm{d}} \xi_i\end{matrix}\right\}_{i\in \mathrm{N}'} \qquad \begin{matrix}\pi\\ \eta_1,\dots,\eta_n,\eta',\xi_1,\dots,\xi_{n'}\vdash_{\mathrm{c}}\eta\end{matrix}}{\Gamma_1,\dots,\Gamma_n,\Gamma,\gamma_1,\gamma_2,\Gamma'',\Gamma'_1,\dots,\Gamma'_{n'}\vdash_{\mathrm{d}}\eta^\circ}(-)^\circ}{\Gamma_1,\dots,\Gamma_n,\Gamma,\Gamma',\Gamma'',\Gamma'_1,\dots,\Gamma'_{n'}\vdash_{\mathrm{d}}\eta^\circ}\circ\varepsilon$$

Figure 5.6: Example of a conversion

in such an elimination rule the formula occurring in the succedent of its minor premise(s) is not related at all to the formula being eliminated: both the $\circ\,\varepsilon$ and the $\epsilon\,\varepsilon$ rules are examples of sum-like elimination rules. A commuting conversion is then simply the permutation of a sum-like elimination rule with another rule in the derivation. An example involving linear instances of the $\circ\,\varepsilon$ rule is:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash_x \gamma_1 \circ \gamma_2} \qquad \dfrac{\pi_1}{\Gamma_1, \gamma_1, \gamma_2, \Gamma_1' \vdash_x \gamma_1' \circ \gamma_2'}}{\Gamma_1, \Gamma, \Gamma_1' \vdash_x \gamma_1' \circ \gamma_2'}\circ\varepsilon \qquad \dfrac{\pi_2}{\Gamma_2, \gamma_1', \gamma_2', \Gamma_2' \vdash_x \gamma}}{\Gamma_2, \Gamma_1, \Gamma, \Gamma_1', \Gamma_2', \vdash_x \gamma}\circ\varepsilon$$

converts to

$$\dfrac{\dfrac{\pi}{\Gamma, \vdash_x \gamma_1 \circ \gamma_2} \qquad \dfrac{\dfrac{\pi_1}{\Gamma_1, \gamma_1, \gamma_2, \Gamma_1' \vdash_x \gamma_1' \circ \gamma_2'} \qquad \dfrac{\pi_2}{\Gamma_2, \gamma_1', \gamma_2', \Gamma_2' \vdash_x \gamma}}{\Gamma_2, \Gamma_1, \gamma_1, \gamma_2, \Gamma_1', \Gamma_2' \vdash_x \gamma}\circ\varepsilon}{\Gamma_2, \Gamma_1, \Gamma, \Gamma_1', \Gamma_2' \vdash_x \gamma}\circ\varepsilon$$

Another example, involving linear instances of the $\circ\,\varepsilon$ and $(-)^{\circ}$ rules, is given in figure 5.6. Note that it induces the equation exemplified in the proof of lemma 5.1.9.

5.3 Discussion and final remarks

We have presented a type theory for the basic causal-deontic fragment of the logic for pragmatics and shown that its terms and equations in context are motivated by the rules and proof transformations of the sequent calculus system given in chapter 3. It is worth mentioning that, since we have introduced the natural deduction system for the basic causal-deontic fragment rather as a tool than as a proper logical formalism on its own, we have only given some hints on how one can actually get strong normalization and confluence results.

Chapter 6

On a semantics of proofs

We come now to the main contribution of this thesis and introduce a categorical semantics for the basic causal-deontic fragment of the pragmatic language. Since, in such a semantics, distinct terms or derivations get distinct denotations we have that categorical models provide a semantics of proofs. In other words, non-degenerate categorical models identify only the morphisms that correspond to the terms occurring in the lefthand side and the righthand side of one of the equations in figures 5.2 or 5.3. Note that this is an immediate consequence of the soundness and completeness results provided in this chapter. Furthermore we also consider algebraic models, a degenerate instance of the categorical semantics, and use them to illustrate how the Kripke models introduced in chapter 4 fit into the general framework.

6.1 Some preliminary definitions

Before we can provide a formal definition of the categorical model for the basic causal-deontic system we need to set up notation. First of all, we introduce the concept of causal-deontic frame (see definition 6.1.8) which allows us to interpret all the rules except for the $(-)^{\circ}$ rule. It exploits the well known relationship between the multiplicative fragment of linear logic and symmetric monoidal closed categories. The building blocks for its definition are given by the following concepts which provide means to take care of the structural rule of contraction, restricted forms of implication, and multiple entailment relations: relevant categories (see definition 6.1.2), restricted closure (see definition 6.1.4), and relevant functors (see definition 6.1.7). The categorical structures needed for the interpretation of the $(-)^{\circ}$ rule are the topic of the remainder of this section (see remark 6.1.11): let us only point out here that the rule is interpreted as a morphism of indexed categories, i.e. a certain natural transformation.

Definition 6.1.1 A *symmetric monoidal category* $(\mathbb{M}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau)$ consists of a category $\mathbb{M}$ together with a functor $\otimes: \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{M}$, an object I of $\mathbb{M}$, and four natural isomorphisms $\alpha_{X,Y,Z}: X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$, $\lambda_X: I \otimes X \longrightarrow X$, $\rho_X: X \otimes I \longrightarrow X$, $\tau_{X,Y}: X \otimes Y \longrightarrow Y \otimes X$ such that $\lambda_I = \rho_I: I \otimes I \longrightarrow I$ and the following five diagrams commute for all objects X, Y, Z, and U of $\mathbb{M}$.

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes U)) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes \alpha"'] & (X \otimes Y) \otimes (Z \otimes U) \arrow[r, "\alpha"] & ((X \otimes Y) \otimes Z) \otimes U \\
X \otimes ((Y \otimes Z) \otimes U) \arrow[rr, "\alpha"'] & & (X \otimes (Y \otimes Z)) \otimes U \arrow[u, "\alpha \otimes \mathrm{id}"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (I \otimes Y) \arrow[d, "\alpha"'] \arrow[dr, "\mathrm{id} \otimes \lambda"] & \\
(X \otimes I) \otimes Y \arrow[r, "\rho \otimes \mathrm{id}"'] & X \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[d, "\tau"'] \arrow[dr, equal] & \\
Y \otimes X \arrow[r, "\tau"'] & X \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes I \arrow[d, "\tau"'] \arrow[dr, "\rho"] & \\
I \otimes X \arrow[r, "\lambda"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes \tau"'] & (X \otimes Y) \otimes Z \arrow[r, "\tau"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\
X \otimes (Z \otimes Y) \arrow[r, "\tau"'] & (X \otimes Z) \otimes Y \arrow[r, "\tau \otimes \mathrm{id}"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

Definition 6.1.2 A *relevant category* $(\mathbb{R}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$ consists of a symmetric monoidal category $(\mathbb{R}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau)$ together with a natural transformation $\delta_X: X \longrightarrow X \otimes X$ such that the following three diagrams commute for all objects X of $\mathbb{R}$.

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, equal] & X \otimes X \arrow[r, "\mathrm{id} \otimes \delta"] & X \otimes (X \otimes X) \arrow[d, "\alpha"] \\
X \arrow[r, "\delta"'] & X \otimes X \arrow[r, "\delta \otimes \mathrm{id}"'] & (X \otimes X) \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[d, "\delta"'] \arrow[dr, equal] & \\
I \otimes I \arrow[r, "\lambda = \rho"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "\delta"'] \arrow[dr, "\delta"] & \\
X \otimes X \arrow[r, "\tau"'] & X \otimes X
\end{tikzcd}$$

Remark 6.1.3 Relevant categories are also known as symmetric monoidal categories with diagonals (see for instance [Jac93] and [Jac94]), and they provide means to interpret the sequents of the basic causal-deontic fragment. Indeed, the key idea is that a term in context $\Gamma \rhd_x \mathtt{M}{:}\,\gamma$, i.e. a sequent of the form $\mathtt{u}_1{:}\,\gamma_1, \ldots, \mathtt{u}_n{:}\,\gamma_n \rhd_x \mathtt{M}{:}\,\gamma$, is interpreted as a morphism $[\![\mathtt{M}]\!]{:}\,[\![\Gamma]\!] \longrightarrow [\![\gamma]\!]$, i.e. a morphism $[\![\mathtt{M}]\!]{:}\,[\![\gamma_1]\!] \otimes \cdots \otimes [\![\gamma_n]\!] \longrightarrow [\![\gamma]\!]$, of some relevant category $\mathbb{X}$. $[\![\mathtt{M}]\!]$ is generally defined by induction and, in the case of the implication-free intuitionistic fragment of multiplicative linear logic, we have that $[\![\mathtt{u}]\!] = \mathrm{id}_{[\![\gamma]\!]}$, $[\![*]\!] = \mathrm{id}_{[\![\epsilon]\!]} = \mathrm{id}_{\mathrm{I}}$, $[\![\mathtt{M}\cdot\mathtt{N}]\!] = [\![\mathtt{M}]\!] \otimes [\![\mathtt{N}]\!]$, and $[\![\mathtt{let\ N\ be}\ p\ \mathtt{in\ M}]\!] = [\![\mathtt{N}]\!] \otimes \mathrm{id}; [\![\mathtt{M}]\!]$ (see for instance [Bie94]). Remember that in the last case p stands for either $\mathtt{v}_1 \cdot \mathtt{v}_2$ or $*$: in the former case we have that $[\![\mathtt{N}]\!] \otimes \mathrm{id}; [\![\mathtt{M}]\!]$ is a morphism $[\![\Gamma_1]\!] \otimes [\![\Gamma_2]\!] \longrightarrow [\![\gamma_1]\!] \otimes [\![\gamma_2]\!] \otimes [\![\Gamma_2]\!] \longrightarrow [\![\gamma]\!]$ of $\mathbb{R}$; in the latter case that it is a morphism $[\![\Gamma_1]\!] \otimes [\![\Gamma_2]\!] \longrightarrow [\![\epsilon]\!] \otimes [\![\Gamma_2]\!] \cong [\![\Gamma_2]\!] \longrightarrow [\![\gamma]\!]$ of $\mathbb{R}$. Diagonals provide means to get several instances of an object and thus allow one to model the implicit contractions that occur in the term formation rules of figure 5.1 with more than one premise: in these cases we precompose the interpretation with a morphism $[\![\Gamma]\!] \otimes [\![\Gamma_1]\!] \otimes [\![\Gamma_2]\!] \longrightarrow ([\![\Gamma]\!] \otimes [\![\Gamma_1]\!]) \otimes ([\![\Gamma]\!] \otimes [\![\Gamma_2]\!])$ obtained by means of δ and the other canonical morphisms.

Definition 6.1.4 Given a relevant category $\mathbb{R} = (\mathbb{R}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$, a relevant subcategory $\mathbb{S}$ of $\mathbb{R}$, and a relevant subcategory $\mathbb{T}$ of $\mathbb{S}$ we say that $\mathbb{S}$ is *closed with respect to* $\mathbb{T}$ if, for all objects T of $\mathbb{T}$, the functor $(-) \otimes \mathrm{T}{:}\,\mathbb{R} \longrightarrow \mathbb{R}$ has a right adjoint $\mathrm{T} \multimap (-){:}\,\mathbb{S} \longrightarrow \mathbb{S}$ in $\mathbb{R}$, i.e. if for all objects R of $\mathbb{R}$, S of $\mathbb{S}$, and T of $\mathbb{T}$ we have that $\mathbb{R}(\mathrm{R} \otimes \mathrm{T}, \mathrm{S}) \cong \mathbb{R}(\mathrm{R}, \mathrm{T} \multimap \mathrm{S})$.

Remark 6.1.5 The standard definition of closure can be obtained as a special case of the above definition: one simply imposes that $\mathbb{T} = \mathbb{S} = \mathbb{R}$. For the purpose of this chapter it would suffice to assume that $\mathbb{S} = \mathbb{R}$. However, when we deal with hypothetical norms in the next chapter we shall see an example where this is not the case. Still another way to express the above definition goes as follows (see for instance [Mac98]): for all objects S of $\mathbb{S}$ and T of $\mathbb{T}$ there exists an object $\mathrm{T} \multimap \mathrm{S}$ of $\mathbb{S}$ together with a map $\mathrm{app}{:}\,(\mathrm{T} \multimap \mathrm{S}) \otimes \mathrm{T} \longrightarrow \mathrm{S}$ of $\mathbb{S}$ such that for every object R of $\mathbb{R}$ and every morphism $\mathrm{f}{:}\,\mathrm{R} \otimes \mathrm{T} \longrightarrow \mathrm{S}$ of $\mathbb{R}$ there exists a unique morphism $\mathrm{cur}(\mathrm{f}){:}\,\mathrm{R} \longrightarrow (\mathrm{T} \multimap \mathrm{S})$ of $\mathbb{R}$ that satisfies $\mathrm{cur}(\mathrm{f}) \otimes \mathrm{id}; \mathrm{app} = \mathrm{f}$. We can thus extend the above definition of $[\![\mathtt{M}]\!]$ with the following two cases for (restricted) linear implication: $[\![\lambda \mathrm{u}.\mathtt{M}]\!] = \mathrm{cur}([\![\mathtt{M}]\!])$ and $[\![(\mathtt{M})\mathtt{N}]\!] = [\![\mathtt{M}]\!] \otimes [\![\mathtt{N}]\!]; \mathrm{app}$. The relevant counterpart of the latter case can again be obtained by precomposing with the canonical morphism $[\![\Gamma]\!] \otimes [\![\Gamma_1]\!] \otimes [\![\Gamma_2]\!] \longrightarrow ([\![\Gamma]\!] \otimes [\![\Gamma_1]\!]) \otimes ([\![\Gamma]\!] \otimes [\![\Gamma_2]\!])$ (compare for instance with remark 6.1.3).

Definition 6.1.6 A *symmetric monoidal functor* $(\mathcal{F}, \mathrm{m}, \mathrm{n}){:}\,\mathbb{M} \longrightarrow \mathbb{M}'$ between two symmetric monoidal categories $\mathbb{M} = (\mathbb{M}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau)$ and $\mathbb{M}' = (\mathbb{M}', \otimes', \mathrm{I}', \alpha', \lambda', \rho', \tau')$ consists of a functor $\mathcal{F}: \mathbb{M} \longrightarrow \mathbb{M}'$ together with a natural transformation $\mathrm{m}_{\mathrm{X,Y}}: \mathcal{F}\mathrm{X} \otimes' \mathcal{F}\mathrm{Y} \longrightarrow \mathcal{F}(\mathrm{X} \otimes \mathrm{Y})$ and a morphism $\mathrm{n}: \mathrm{I}' \longrightarrow \mathcal{F}\mathrm{I}$ such that, for all objects X, Y, and Z of $\mathbb{M}$, the following four diagrams commute in $\mathbb{M}'$.

$$\begin{tikzcd}
(\mathcal{F}\mathrm{X} \otimes' \mathcal{F}\mathrm{Y}) \otimes' \mathcal{F}\mathrm{Z} \arrow[r, "\mathrm{m} \otimes' \mathrm{id}"] & \mathcal{F}(\mathrm{X} \otimes \mathrm{Y}) \otimes' \mathcal{F}\mathrm{Z} \arrow[r, "\mathrm{m}"] & \mathcal{F}((\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z}) \\
\mathcal{F}\mathrm{X} \otimes' (\mathcal{F}\mathrm{Y} \otimes' \mathcal{F}\mathrm{Z}) \arrow[u, "\alpha'"] \arrow[r, "\mathrm{id} \otimes' \mathrm{m}"'] & \mathcal{F}\mathrm{X} \otimes' \mathcal{F}(\mathrm{Y} \otimes \mathrm{Z}) \arrow[r, "\mathrm{m}"'] & \mathcal{F}(\mathrm{X} \otimes (\mathrm{Y} \otimes \mathrm{Z})) \arrow[u, "\mathcal{F}(\alpha)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}\mathrm{I} \otimes' \mathcal{F}\mathrm{X} \arrow[r, "\mathrm{m}"] & \mathcal{F}(\mathrm{I} \otimes \mathrm{X}) \arrow[d, "\mathcal{F}(\lambda)"] \\
\mathrm{I}' \otimes' \mathcal{F}\mathrm{X} \arrow[u, "\mathrm{n} \otimes' \mathrm{id}"] \arrow[r, "\lambda'"'] & \mathcal{F}\mathrm{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}\mathrm{X} \otimes' \mathcal{F}\mathrm{I} \arrow[r, "\mathrm{m}"] & \mathcal{F}(\mathrm{X} \otimes \mathrm{I}) \arrow[d, "\mathcal{F}(\rho)"] \\
\mathcal{F}\mathrm{X} \otimes' \mathrm{I}' \arrow[u, "\mathrm{id} \otimes' \mathrm{n}"] \arrow[r, "\rho'"'] & \mathcal{F}\mathrm{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}\mathrm{X} \otimes' \mathcal{F}\mathrm{Y} \arrow[r, "\mathrm{m}"] \arrow[d, "\tau'"'] & \mathcal{F}(\mathrm{X} \otimes \mathrm{Y}) \arrow[d, "\mathcal{F}(\tau)"] \\
\mathcal{F}\mathrm{Y} \otimes' \mathcal{F}\mathrm{X} \arrow[r, "\mathrm{m}"'] & \mathcal{F}(\mathrm{Y} \otimes \mathrm{X})
\end{tikzcd}$$

$(\mathcal{F}, \mathrm{m}, \mathrm{n}): \mathbb{M} \longrightarrow \mathbb{M}'$ is called *strong* if m is a natural isomorphism and n an isomorphism; it is called *strict* if both m and n are identities.

Definition 6.1.7 A *relevant functor* $(\mathcal{F}, \mathrm{m}, \mathrm{n}): \mathbb{R} \longrightarrow \mathbb{R}'$ between two relevant categories $\mathbb{R} = (\mathbb{R}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$ and $\mathbb{R}' = (\mathbb{R}', \otimes', \mathrm{I}', \alpha', \lambda', \rho', \tau', \delta')$ is a symmetric monoidal functor $(\mathcal{F}, \mathrm{m}, \mathrm{n}): \mathbb{R} \longrightarrow \mathbb{R}'$ such that the following diagram commutes for all objects X of $\mathbb{R}$.

$$\begin{tikzcd}
\mathcal{F}\mathrm{X} \arrow[d, "\delta'"'] \arrow[dr, "\mathcal{F}(\delta)"] & \\
\mathcal{F}\mathrm{X} \otimes' \mathcal{F}\mathrm{X} \arrow[r, "\mathrm{m}"'] & \mathcal{F}(\mathrm{X} \otimes \mathrm{X})
\end{tikzcd}$$

$(\mathcal{F}, \mathrm{m}, \mathrm{n}): \mathbb{R} \longrightarrow \mathbb{R}'$ is called *strong* if m is a natural isomorphism and n an isomorphism; it is called *strict* if both m and n are identities.

Definition 6.1.8 A *causal-deontic frame* $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{O})$ consists of three relevant categories

$$\mathbb{A} = (\mathbb{A}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta) \qquad \mathbb{C} = (\mathbb{C}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta) \qquad \mathbb{D} = (\mathbb{D}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$$

together with three relevant functors

$$\mathcal{J}_{\mathrm{ac}} = (\mathcal{J}_{\mathrm{ac}}, \iota_2, \iota_0): \mathbb{A} \longrightarrow \mathbb{C} \qquad \mathcal{J}_{\mathrm{cd}} = (\mathcal{J}_{\mathrm{cd}}, \iota_2, \iota_0): \mathbb{C} \longrightarrow \mathbb{D} \qquad \mathcal{O} = (\mathcal{O}, \theta_2, \theta_0): \mathbb{A} \longrightarrow \mathbb{D}$$

such that both $\mathcal{J}_{\mathrm{ac}}$ and $\mathcal{J}_{\mathrm{cd}}$ are strict, and $\mathbb{C}$ is closed with respect to its subcategory $\mathcal{J}_{\mathrm{ac}}\mathbb{A}$.

Remark 6.1.9 It is worth mentioning that since the functor $\mathcal{O}\colon \mathbb{A} \longrightarrow \mathbb{D}$ provides means to interpret the $(-)^{\circ}$ operator it is crucial that it is not simply defined as $\mathcal{J}_{\mathrm{ac}};\mathcal{J}_{\mathrm{cd}}\colon \mathbb{A} \longrightarrow \mathbb{D}$, the composition of the two functors $\mathcal{J}_{\mathrm{ac}}$ and $\mathcal{J}_{\mathrm{cd}}$. Both $\mathcal{J}_{\mathrm{ac}}$ and $\mathcal{J}_{\mathrm{cd}}$ are required to be strict relevant functors whereas $\mathcal{O}$ does not need to satisfy this property. The assumption that both $\mathcal{J}_{\mathrm{ac}}$ and $\mathcal{J}_{\mathrm{cd}}$ are strict, i.e. that ι_2 and ι_0 are actually identities, is justified by the fact that we think of both of them as inclusion functors (compare also with remark 3.1.2). Note however that this is not a severe limitation: the more general case where both $\mathcal{J}_{\mathrm{ac}}$ and $\mathcal{J}_{\mathrm{cd}}$ are required to be strong relevant functors can be obtained in a straightforward fashion.

Remark 6.1.10 From now on we shall make extended use of the following concept: a category without identities consists of a collection of objects and a collection of morphism such that the composition of morphisms is associative. In other words, we drop the requirement that there is an identity morphism satisfying the identity law from the usual definition of category. We follow [MBCB02] and call categories without identities semi-categories.[1] Arrows between semi-categories or semi-functors are simply composition-preserving maps. Note that semi-categories and semi-functors form a category which we denote with SCat.

Remark 6.1.11 The remainder of this section is devoted to the categorical structures needed for the interpretation of the modal rule. To get a rationale for its semantics let us consider the following instance

$$\frac{\eta_1, \xi \Rightarrow_{\mathrm{c}} \eta_2}{\eta_1^{\circ}, \xi \Rightarrow_{\mathrm{d}} \eta_2^{\circ}}$$

of the $(-)^{\circ}$ rule in sequent calculus notation. First of all, we shall assume that some specific causal-deontic frame $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{O})$ is given. Clearly, in the categorical model we need to interpret the premise of the rule as a morphism in the category $\mathbb{C}$ and the conclusion of the rule as a morphism in $\mathbb{D}$. The definition of causal-deontic frame naturally suggests the following interpretations:

$$\llbracket \eta_1, \xi \Rightarrow_{\mathrm{c}} \eta_2 \rrbracket = \mathcal{J}_{\mathrm{ac}}\llbracket \eta_1 \rrbracket \otimes \llbracket \xi \rrbracket \longrightarrow \mathcal{J}_{\mathrm{ac}}\llbracket \eta_2 \rrbracket \quad (\text{in } \mathbb{C})$$
$$\llbracket \eta_1^{\circ}, \xi \Rightarrow_{\mathrm{d}} \eta_2^{\circ} \rrbracket = \mathcal{O}\llbracket \eta_1 \rrbracket \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \xi \rrbracket \longrightarrow \mathcal{O}\llbracket \eta_2 \rrbracket \quad (\text{in } \mathbb{D})$$

But how do we model the passage from the premise to the conclusion within this context? Let us observe that: (a) both interpretations have to live in the same environment, so that they can

[1] Let us make two remarks about terminology: first, in [MB03] the term *multiplicative graph* is used instead of semi-category; second, the term *semicategory* is sometimes also used to denote a different concept (see, for instance [SH00]) not relevant to the purpose of this thesis.

interact in some specific way; (b) the above instance of the $(-)^{\circ}$ rule depends on the parameter ξ (see also remark 5.1.3 for a natural deduction formulation of the $(-)^{\circ}$ rule with explicit parameters). A natural way to take care of these aspects is given by split indexed categories: these are functors from a category $\mathbb{B}^{\mathrm{op}}$ to $\mathbf{Cat}$, the category of small categories (see for instance [Jac99]). Since $[\![\xi]\!]$ is an object of $\mathbb{C}$ (and thus of $\mathbb{C}^{\mathrm{op}}$) we are looking for suitable functors from $\mathbb{C}^{\mathrm{op}}$ to $\mathbf{Cat}$ that have the above interpretations among the morphisms of its image category. The obvious choice is to assign categories with morphisms of the form $\mathcal{J}_{\mathrm{ac}}[\![\eta_1]\!] \otimes [\![\xi]\!] \longrightarrow \mathcal{J}_{\mathrm{ac}}[\![\eta_2]\!]$ and $\mathcal{O}[\![\eta_1]\!] \otimes \mathcal{J}_{\mathrm{cd}}[\![\xi]\!] \longrightarrow \mathcal{O}[\![\eta_2]\!]$ to an object $[\![\xi]\!]$ of $\mathbb{C}^{\mathrm{op}}$ but then we do not have identities in the image categories, i.e. the images of such functors are only semi-categories. Therefore, if we generalise the above situation and replace $\mathbf{Cat}$ by $\mathbf{SCat}$, we can define functors $\mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{SCat}$ (see definitions 6.1.12 and 6.1.13 below) that satisfy the required properties. It is worth pointing out that the morphisms in the image semi-category of the latter functor are actually of the form $[\![\gamma_1]\!] \otimes \mathcal{J}_{\mathrm{cd}}[\![\xi]\!] \longrightarrow [\![\gamma_2]\!]$. Nevertheless, within this framework the $(-)^{\circ}$ rule can be modelled as a natural transformation between the two functors, i.e. as a morphism of generalised split indexed categories that maps a morphism of the form $\mathcal{J}_{\mathrm{ac}}[\![\eta_1]\!] \otimes [\![\xi]\!] \longrightarrow \mathcal{J}_{\mathrm{ac}}[\![\eta_2]\!]$ to the morphism $\mathcal{O}[\![\eta_1]\!] \otimes \mathcal{J}_{\mathrm{cd}}[\![\xi]\!] \longrightarrow \mathcal{O}[\![\eta_2]\!]$ (see definition 6.1.16 below).

Definition 6.1.12 The *restriction functor* $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}\colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{SCat}$ is defined as follows. It maps

1. an object C of $\mathbb{C}$ to the semi-category $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C})$ that has
   (a) objects $\mathcal{J}_{\mathrm{ac}}\mathrm{A}$ of $\mathbb{C}$ as objects,
   (b) morphisms $\mathcal{J}_{\mathrm{ac}}\mathrm{A}_1 \otimes \mathrm{C} \longrightarrow \mathcal{J}_{\mathrm{ac}}\mathrm{A}_2$ of $\mathbb{C}$ as morphisms;
2. a morphism $\mathrm{f}\colon \mathrm{C}_1 \longrightarrow \mathrm{C}_2$ of $\mathbb{C}$ to the semi-functor $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{f})\colon \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_2) \longrightarrow \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_1)$ which, in turn, maps
   (a) an object $\mathcal{J}_{\mathrm{ac}}\mathrm{A}$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_2)$ to the object $\mathcal{J}_{\mathrm{ac}}\mathrm{A}$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_1)$,
   (b) a morphism $\mathrm{g}\colon \mathcal{J}_{\mathrm{ac}}\mathrm{A}_1 \otimes \mathrm{C}_2 \longrightarrow \mathcal{J}_{\mathrm{ac}}\mathrm{A}_2$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_2)$ to the following morphism of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(\mathrm{C}_1)$.

$$\mathcal{J}_{\mathrm{ac}}\mathrm{A}_1 \otimes \mathrm{C}_1 \xrightarrow{\mathrm{id} \otimes \mathrm{f}} \mathcal{J}_{\mathrm{ac}}\mathrm{A}_1 \otimes \mathrm{C}_2 \xrightarrow{\mathrm{g}} \mathcal{J}_{\mathrm{ac}}\mathrm{A}_2$$

Definition 6.1.13 The *expansion functor* $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}\colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{SCat}$ is defined as follows. It maps

1. an object C of $\mathbb{C}$ to the semi-category $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(\mathrm{C})$ that has

(a) objects D of $\mathbb{D}$ as objects,

(b) morphisms $D_1 \otimes \mathcal{J}_{cd}C \longrightarrow D_2$ of $\mathbb{D}$ as morphisms;

2. a morphism $f: C_1 \longrightarrow C_2$ of $\mathbb{C}$ to the semi-functor $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(f): \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_2) \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_1)$ which, in turn, maps

   (a) an object D of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_2)$ to the object D of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_1)$,

   (b) a morphism $g: D_1 \otimes \mathcal{J}_{cd}C_2 \longrightarrow D_2$ of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_2)$ to the following morphism of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_1)$.

$$D_1 \otimes \mathcal{J}_{cd}C_1 \xrightarrow{\mathrm{id} \otimes \mathcal{J}_{cd}f} D_1 \otimes \mathcal{J}_{cd}C_2 \xrightarrow{g} D_2$$

Lemma 6.1.14 $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}: \mathbb{C}^{op} \longrightarrow \mathbf{SCat}$ and $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}: \mathbb{C}^{op} \longrightarrow \mathbf{SCat}$ are indeed functors.

Proof Easy, since preservation of identities and compositions is inherited from $\mathbb{C}$. □

Lemma 6.1.15 Given an object C of $\mathbb{C}$, $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C)$ and $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$ are indeed semi-categories.

Proof We provide only the proof for $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C)$, the proof for $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$ being analogous. Given morphisms

$$f: \mathcal{J}_{ac}A_1 \otimes C \longrightarrow \mathcal{J}_{ac}A' \qquad f': \mathcal{J}_{ac}A' \otimes C \longrightarrow \mathcal{J}_{ac}A_2$$

of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C)$, their composition is obtained by precomposing f' with the morphism

$$\mathcal{J}_{ac}A_1 \otimes C \xrightarrow{\mathrm{id} \otimes \delta} \mathcal{J}_{ac}A_1 \otimes (C \otimes C) \xrightarrow{\alpha} (\mathcal{J}_{ac}A_1 \otimes C) \otimes C \xrightarrow{f \otimes \mathrm{id}} \mathcal{J}_{ac}A' \otimes C$$

of $\mathbb{C}$. That composition is associative follows by a straightforward calculation. □

Definition 6.1.16 A *causal-deontic transformation* $\vartheta: \mathcal{R}^{\mathbb{A}}_{\mathbb{C}} \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}$ is a natural transformation $\vartheta_C: \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C) \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$ that maps

1. an object $\mathcal{J}_{ac}A$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C)$ to the object $\mathcal{O}A$ of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$,

2. a morphism $\mathcal{J}_{ac}A_1 \otimes C \longrightarrow \mathcal{J}_{ac}A_2$ of $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C)$ to the morphism $\mathcal{O}A_1 \otimes \mathcal{J}_{cd}C \longrightarrow \mathcal{O}A_2$ of $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$.

Remark 6.1.17 It is worth mentioning that since, for any object C of $\mathbb{C}$, both $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C)$ and $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$ are semi-categories $\vartheta_C: \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C) \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C)$ is actually a semi-functor. At the level of morphisms we have that the naturality square

$$\begin{tikzcd} \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C_2) \arrow[r, "\vartheta_{C_2}"] \arrow[d, "\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(f)"'] & \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_2) \arrow[d, "\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(f)"] \\ \mathcal{R}^{\mathbb{A}}_{\mathbb{C}}(C_1) \arrow[r, "\vartheta_{C_1}"'] & \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}(C_1) \end{tikzcd}$$

boils down to the commuting diagram

$$\begin{tikzcd} \mathcal{O}A_1 \otimes \mathcal{J}_{cd}C_1 \arrow[r, "\vartheta_{C_1}(\mathrm{id} \otimes f; g)"] \arrow[d, "\mathrm{id} \otimes \mathcal{J}_{cd}(f)"'] & \mathcal{O}A_2 \arrow[d, equal] \\ \mathcal{O}A_1 \otimes \mathcal{J}_{cd}C_2 \arrow[r, "\vartheta_{C_2}(g)"'] & \mathcal{O}A_2 \end{tikzcd}$$

where $f\colon C_1 \longrightarrow C_2$ and $g\colon \mathcal{J}_{ac}A_1 \otimes C_2 \longrightarrow \mathcal{J}_{ac}A_2$.

6.2 Soundness and completeness

We are now in the position to provide the definition of the categorical model (see definition 6.2.1 and remark 6.2.3) and thus to establish a semantics of proofs for the basic causal-deontic system. We show that the categorical semantics is sound and complete (see theorems 6.2.5 and 6.2.6) with respect to the type theory provided in the previous chapter.

Definition 6.2.1 A *causal-deontic category* $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{ac}, \mathcal{J}_{cd}, \mathcal{O}, \vartheta)$ is a causal-deontic frame $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{ac}, \mathcal{J}_{cd}, \mathcal{O})$ endowed with a causal-deontic transformation $\vartheta\colon \mathcal{R}^{\mathbb{A}}_{\mathbb{C}} \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}$ such that, for all morphisms $f_1\colon \mathcal{J}_{ac}A_1 \otimes C_1 \longrightarrow \mathcal{J}_{ac}A'$ and $f_2\colon \mathcal{J}_{ac}A' \otimes C_2 \longrightarrow \mathcal{J}_{ac}A_2$ of $\mathbb{C}$, the following coherence diagram commutes in $\mathbb{D}$.

$$\begin{tikzcd} \mathcal{O}A_1 \otimes \mathcal{J}_{cd}(C_1 \otimes C_2) \arrow[rr, "\vartheta_{C_1 \otimes C_2}(\alpha; f_1 \otimes \mathrm{id}; f_2)"] \arrow[d, equal] & & \mathcal{O}A_2 \\ \mathcal{O}A_1 \otimes (\mathcal{J}_{cd}C_1 \otimes \mathcal{J}_{cd}C_2) \arrow[d, "\alpha"'] & & \\ (\mathcal{O}A_1 \otimes \mathcal{J}_{cd}C_1) \otimes \mathcal{J}_{cd}C_2 \arrow[rr, "\vartheta_{C_1}(f_1) \otimes \mathrm{id}"'] & & \mathcal{O}A' \otimes \mathcal{J}_{cd}C_2 \arrow[uu, "\vartheta_{C_2}(f_2)"'] \end{tikzcd}$$

Remark 6.2.2 Note that, whereas the naturality square of remark 6.1.17 provides means to deal with the substitution of parameters in the interpretation of the $(-)^{\circ}$ rule, the above coherence condition allows one to deal with the substitution of objects which are not parameters (see remark 5.1.3 for a natural deduction formulation of the rule with explicit parameters and lemma 6.2.4 below for a categorical reformulation of substitution). It is an open question whether this condition can be made implicit by assuming some suitable structure on the functors $\mathcal{R}^{\mathbb{A}}_{\mathbb{C}}$ and $\mathcal{E}^{\mathbb{D}}_{\mathbb{C}}$, and hence on the causal-deontic transformation $\vartheta\colon \mathcal{R}^{\mathbb{A}}_{\mathbb{C}} \longrightarrow \mathcal{E}^{\mathbb{D}}_{\mathbb{C}}$.

Remark 6.2.3 In order to complete the formal definition of the categorical model we need to explain how types and terms are interpreted in a causal-deontic category. Instead of going into the details we only give some hints. First, the interpretation function $[\![-]\!]$ maps assertive, pure causal, and causal-deontic formulae or types to objects of $\mathbb{A}$, $\mathbb{C}$, and $\mathbb{D}$, respectively:

1. assertive types: $[\![\eta_p]\!] = \mathrm{A} \in obj(\mathbb{A})$, $[\![\epsilon]\!] = \mathrm{I}$, and $[\![\eta_1 \circ \eta_2]\!] = [\![\eta_1]\!] \otimes [\![\eta_2]\!]$;

2. pure causal types: $[\![\eta]\!] = \mathcal{J}_{\mathrm{ac}}[\![\eta]\!]$, $[\![\eta \supset \xi]\!] = \mathcal{J}_{\mathrm{ac}}[\![\eta]\!] \multimap [\![\xi]\!]$, and $[\![\xi_1 \circ \xi_2]\!] = [\![\xi_1]\!] \otimes [\![\xi_2]\!]$;

3. causal-deontic types: $[\![\eta^\circ]\!] = \mathcal{O}[\![\eta]\!]$, $[\![\xi]\!] = \mathcal{J}_{\mathrm{cd}}[\![\xi]\!]$, and $[\![\gamma_1 \circ \gamma_2]\!] = [\![\gamma_1]\!] \otimes [\![\gamma_2]\!]$.

Note that the interpretation $[\![\eta]\!]$ of an assertive type η in $\mathbb{C}$ is obtained as the image along $\mathcal{J}_{\mathrm{ac}}$ of the interpretation of η in $\mathbb{A}$; the interpretation $[\![\xi]\!]$ of a pure causal type ξ in $\mathbb{D}$ follows a similar pattern. Since the intended relevant category, be it either $\mathbb{A}$, $\mathbb{C}$, or $\mathbb{D}$, will always be clear from the context this overloading of notation is harmless. Second, since we have already outlined in remark 6.1.3 and 6.1.5 how the multiplicative fragment of intuitionistic linear logic with contractions and a restricted form of implication can be interpreted by means of relevant categories with restricted closure, we limit ourself to spell out the interpretations of the terms for the $(\mathtt{a/c})$, $(\mathtt{c/d})$, and $(-)^\circ$ rules of figure 5.1: these are given in figure 6.1 where, for simplicity, we have omitted the implicit contractions.

Lemma 6.2.4 Given $\Gamma \rhd_{\mathrm{x}} \mathtt{N}\colon \gamma'$ and $\Gamma_1, \mathtt{u}\colon \gamma', \Gamma_2 \rhd_{\mathrm{x}} \mathtt{M}\colon \gamma$, we have that $[\![\mathtt{M}[\mathtt{N}/\mathtt{u}]]\!]$ is given by the morphism

$$[\![\Gamma_1]\!] \otimes [\![\Gamma]\!] \otimes [\![\Gamma_2]\!] \xrightarrow{\mathrm{id} \otimes [\![\mathtt{N}]\!] \otimes \mathrm{id}} [\![\Gamma_1]\!] \otimes [\![\gamma']\!] \otimes [\![\Gamma_2]\!] \xrightarrow{[\![\mathtt{M}]\!]} [\![\gamma]\!]$$

of $\mathbb{X}$ where, depending on the inference relation $\rhd_{\mathrm{x}}$, $\mathbb{X}$ is either one of the categories $\mathbb{A}$, $\mathbb{C}$, or $\mathbb{D}$.

Proof By induction on the structure of the term $\mathtt{M}$ (compare with lemma 5.1.5). $\square$

$$\frac{\Xi_1 \rhd_{\mathsf{c}} \mathrm{M}_1 : \eta_1 \quad \cdots \quad \Xi_n \rhd_{\mathsf{c}} \mathrm{M}_n : \eta_n \qquad \mathrm{u}_1 : \eta_1, \ldots, \mathrm{u}_n : \eta_n \rhd_{\mathsf{a}} \mathrm{M} : \eta}{\Xi_1, \ldots, \Xi_n \rhd_{\mathsf{c}} \{\mathsf{a} \mapsto \mathsf{c}\}[\mathrm{M}_1, \ldots, \mathrm{M}_n / \mathrm{u}_1, \ldots, \mathrm{u}_n \mid \mathrm{M}] : \eta} \ (\mathsf{a}/\mathsf{c})$$

$$\begin{tikzcd}
\llbracket \Xi_1 \rrbracket \otimes \cdots \otimes \llbracket \Xi_n \rrbracket \arrow[r, "{\llbracket \{\mathsf{a} \mapsto \mathsf{c}\}[\mathrm{M}_1, \ldots, \mathrm{M}_n / \mathrm{u}_1, \ldots, \mathrm{u}_n \mid \mathrm{M}] \rrbracket}"] \arrow[d, "\llbracket \mathrm{M}_1 \rrbracket \otimes \cdots \otimes \llbracket \mathrm{M}_n \rrbracket"'] \arrow[dr, phantom, "=_{def}"] & \mathcal{J}_{\mathsf{ac}} \llbracket \eta \rrbracket \\
\mathcal{J}_{\mathsf{ac}} \llbracket \eta_1 \rrbracket \otimes \cdots \otimes \mathcal{J}_{\mathsf{ac}} \llbracket \eta_n \rrbracket \arrow[r, equal] & \mathcal{J}_{\mathsf{ac}}(\llbracket \eta_1 \rrbracket \otimes \cdots \otimes \llbracket \eta_n \rrbracket) \arrow[u, "\mathcal{J}_{\mathsf{ac}} \llbracket \mathrm{M} \rrbracket"']
\end{tikzcd}$$

$$\frac{\Gamma_1 \rhd_{\mathsf{d}} \mathrm{M}_1 : \xi_1 \quad \cdots \quad \Gamma_n \rhd_{\mathsf{d}} \mathrm{M}_n : \xi_n \qquad \mathrm{u}_1 : \xi_1, \ldots, \mathrm{u}_n : \xi_n \rhd_{\mathsf{c}} \mathrm{M} : \xi}{\Gamma_1, \ldots, \Gamma_n \rhd_{\mathsf{d}} \{\mathsf{c} \mapsto \mathsf{d}\}[\mathrm{M}_1, \ldots, \mathrm{M}_n / \mathrm{u}_1, \ldots, \mathrm{u}_n \mid \mathrm{M}] : \xi} \ (\mathsf{c}/\mathsf{d})$$

$$\begin{tikzcd}
\llbracket \Gamma_1 \rrbracket \otimes \cdots \otimes \llbracket \Gamma_n \rrbracket \arrow[r, "{\llbracket \{\mathsf{c} \mapsto \mathsf{d}\}[\mathrm{M}_1, \ldots, \mathrm{M}_n / \mathrm{u}_1, \ldots, \mathrm{u}_n \mid \mathrm{M}] \rrbracket}"] \arrow[d, "\llbracket \mathrm{M}_1 \rrbracket \otimes \cdots \otimes \llbracket \mathrm{M}_n \rrbracket"'] \arrow[dr, phantom, "=_{def}"] & \mathcal{J}_{\mathsf{cd}} \llbracket \xi \rrbracket \\
\mathcal{J}_{\mathsf{cd}} \llbracket \xi_1 \rrbracket \otimes \cdots \otimes \mathcal{J}_{\mathsf{cd}} \llbracket \xi_n \rrbracket \arrow[r, equal] & \mathcal{J}_{\mathsf{cd}}(\llbracket \xi_1 \rrbracket \otimes \cdots \otimes \llbracket \xi_n \rrbracket) \arrow[u, "\mathcal{J}_{\mathsf{cd}} \llbracket \mathrm{M} \rrbracket"']
\end{tikzcd}$$

$$\frac{\{\Gamma_i \rhd_{\mathsf{d}} \mathrm{M}_i : \eta_i^{\circ}\}_{i \in \mathrm{N}} \qquad \{\Gamma'_i \rhd_{\mathsf{d}} \mathrm{M}'_i : \xi_i\}_{i \in \mathrm{M}} \qquad \underline{\mathrm{u}} : \vec{\eta}, \underline{\mathrm{u}}' : \Xi \rhd_{\mathsf{c}} \mathrm{M} : \eta}{\Gamma_1, \ldots, \Gamma_n, \Gamma'_1, \ldots, \Gamma'_m \rhd_{\mathsf{d}} \mathsf{prm\_ob}\, [\underline{\mathrm{M}}, \underline{\mathrm{M}}' / \underline{\mathrm{u}}, \underline{\mathrm{u}}' \mid \mathrm{M}] : \eta^{\circ}} \ (-)^{\circ}$$

$$\begin{tikzcd}
(\bigotimes_{i=1}^{n} \llbracket \Gamma_i \rrbracket) \otimes (\bigotimes_{i=1}^{m} \llbracket \Gamma'_i \rrbracket) \arrow[r, "{\llbracket \mathsf{prm_ob}\, [\underline{\mathrm{M}}, \underline{\mathrm{M}}' / \underline{\mathrm{u}}, \underline{\mathrm{u}}' \mid \mathrm{M}] \rrbracket}"] \arrow[d, "\llbracket \underline{\mathrm{M}} \rrbracket \otimes \llbracket \underline{\mathrm{M}}' \rrbracket"'] \arrow[dr, phantom, "=_{def}"] & \mathcal{O} \llbracket \eta \rrbracket \\
(\bigotimes_{i=1}^{n} \mathcal{O} \llbracket \eta_i \rrbracket) \otimes (\bigotimes_{i=1}^{m} \mathcal{J}_{\mathsf{cd}} \llbracket \xi_i \rrbracket) \arrow[r, "\theta_n \otimes \mathrm{id}"'] & \mathcal{O}(\bigotimes_{i=1}^{n} \llbracket \eta_i \rrbracket) \otimes \mathcal{J}_{\mathsf{cd}}(\bigotimes_{i=1}^{m} \llbracket \xi_i \rrbracket) \arrow[u, "\vartheta_{\llbracket \Xi \rrbracket} \llbracket \mathrm{M} \rrbracket"']
\end{tikzcd}$$

Figure 6.1: Terms and interpretations

Theorem 6.2.5 (soundness) Given any causal-deontic category $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{O}, \vartheta)$ together with an interpretation $[\![-]\!]$, if the equation in context $\Gamma \rhd_{\mathrm{x}} \mathtt{M} = \mathtt{M}' \colon \gamma$ is derivable then $[\![\mathtt{M}]\!] = [\![\mathtt{M}']\!]$ holds in $\mathbb{X}$ ($= \mathbb{A}$, $\mathbb{C}$, or $\mathbb{D}$).

Proof We need to show that all equations $[\![\mathtt{M}]\!] = [\![\mathtt{M}']\!]$ induced by the equations given in figures 5.2 and 5.3 hold in the causal-deontic category. We consider only the case of the last equation of figure 5.2: details of the proof are summarised in figures 6.2 and 6.3. Note the fundamental role of the coherence condition provided in definition 6.2.1. One can adapt the proof to the case with implicit contractions by precomposing the left leg and the right leg of the upper square in the commuting diagram of figure 6.2 with an appropriate canonical morphism. □

Theorem 6.2.6 (completeness) For any causal-deontic category $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{O}, \vartheta)$ and for any interpretation $[\![-]\!]$, if $[\![\mathtt{M}]\!] = [\![\mathtt{M}']\!]$ holds in $\mathbb{X}$ ($= \mathbb{A}$, $\mathbb{C}$, or $\mathbb{D}$) for the terms in context $\Gamma \rhd_{\mathrm{x}} \mathtt{M} \colon \gamma$ and $\Gamma \rhd_{\mathrm{x}} \mathtt{M}' \colon \gamma$ then the equation in context $\Gamma \rhd_{\mathrm{x}} \mathtt{M} = \mathtt{M}' \colon \gamma$ is derivable.

Proof This is a straightforward consequence of the following lemma: the key observation is that the interpretation function is simply given by the identity function. □

Lemma 6.2.7 There exists $(\underline{\mathbb{A}}, \underline{\mathbb{C}}, \underline{\mathbb{D}}, \underline{\mathcal{J}_{\mathrm{ac}}}, \underline{\mathcal{J}_{\mathrm{cd}}}, \underline{\mathcal{O}}, \underline{\vartheta})$ such that every morphism f of $\underline{\mathbb{X}}$ ($= \underline{\mathbb{A}}$, $\underline{\mathbb{C}}$, or $\underline{\mathbb{D}}$) is the interpretation of a term in context $\Gamma \rhd_{\mathrm{x}} \mathtt{M} \colon \gamma$.

Proof The so-called syntactic category (or term model) is constructed as follows. Its objects are formulae γ and its morphisms are equivalence classes of natural deduction derivations π of $\gamma_1 \vdash_{\mathrm{x}} \gamma_2$. Each fragment of the natural deduction system induces one of the categories $\underline{\mathbb{A}}$, $\underline{\mathbb{C}}$, and $\underline{\mathbb{D}}$. The following instances

$$\frac{\overline{\eta_1 \vdash_{\mathrm{c}} \eta_1} \qquad \begin{matrix}\pi \\ \eta_1 \vdash_{\mathrm{a}} \eta_2\end{matrix}}{\eta_1 \vdash_{\mathrm{c}} \eta_2} \qquad\qquad \frac{\overline{\xi_1 \vdash_{\mathrm{d}} \xi_1} \qquad \begin{matrix}\pi \\ \xi_1 \vdash_{\mathrm{c}} \xi_2\end{matrix}}{\xi_1 \vdash_{\mathrm{d}} \xi_2}$$

of (a/c) and (c/d) give rise to trivially relevant inclusion functors associated with $\underline{\mathbb{A}} \subseteq_{full} \underline{\mathbb{C}} \subseteq_{full} \underline{\mathbb{D}}$. That $\underline{\mathbb{C}}$ is closed with respect to $\underline{\mathbb{A}}$ is an immediate consequence of the existence of the derived rules:

$$\dfrac{\dfrac{\dfrac{\overline{\eta \vdash_{\mathrm{c}} \eta} \qquad \overline{\xi' \vdash_{\mathrm{c}} \xi'}}{\eta, \xi' \vdash_{\mathrm{c}} \eta \circ \xi'}\circ\mathcal{I} \qquad \begin{matrix}\pi \\ \eta \circ \xi' \vdash_{\mathrm{c}} \xi\end{matrix}}{\eta, \xi' \vdash_{\mathrm{c}} \xi}\,\mathtt{sub}}{\xi' \vdash_{\mathrm{c}} \eta \supset \xi}\supset\mathcal{I}$$

Equation in context

$$
\frac{
\begin{array}{lll}
\{\Gamma_{1i} \rhd_{\mathrm{d}} \mathtt{M}_{1i} \colon \eta^{\circ}_{1i}\}_{i \in N_1} & \{\Gamma'_{1i} \rhd_{\mathrm{d}} \mathtt{M}'_{1i} \colon \xi_{1i}\}_{i \in M_1} & \underline{\mathtt{u}}_1 \colon \vec{\eta}_1, \underline{\mathtt{u}}'_1 \colon \Xi_1 \rhd_{\mathrm{c}} \mathtt{M}' \colon \eta' \\
\{\Gamma_{2i} \rhd_{\mathrm{d}} \mathtt{M}_{2i} \colon \eta^{\circ}_{2i}\}_{i \in N_2} & \{\Gamma'_{2i} \rhd_{\mathrm{d}} \mathtt{M}'_{2i} \colon \xi_{2i}\}_{i \in M_2} & (\underline{\mathtt{u}}_2 \colon \vec{\eta}_2, \mathtt{u} \colon \eta'), \underline{\mathtt{u}}'_2 \colon \Xi_2 \rhd_{\mathrm{c}} \mathtt{M} \colon \eta
\end{array}
}{
\begin{array}{l}
(\underline{\Gamma}_2, \underline{\Gamma}_1), (\underline{\Gamma}'_1, \underline{\Gamma}'_2) \rhd_{\mathrm{d}} \mathtt{prm\_ob}\, [\underline{\mathtt{M}}_2, (\mathtt{prm\_ob}\, [\underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1 / \underline{\mathtt{u}}_1, \underline{\mathtt{u}}'_1 \mid \mathtt{M}']), \underline{\mathtt{M}}'_2 / \underline{\mathtt{u}}_2, \mathtt{u}, \underline{\mathtt{u}}'_2 \mid \mathtt{M}] \\
\qquad = \mathtt{prm\_ob}\, [\underline{\mathtt{M}}_2, \underline{\mathtt{M}}_1, \underline{\mathtt{M}}'_1, \underline{\mathtt{M}}'_2 / \underline{\mathtt{u}}_1, \underline{\mathtt{u}}_2, \underline{\mathtt{u}}'_1, \underline{\mathtt{u}}'_2 \mid \mathtt{M}[\mathtt{M}'/\mathtt{u}]] \colon \eta^{\circ}
\end{array}
}
$$

Commuting diagram

$$
\begin{tikzcd}
(\llbracket \underline{\Gamma}_2 \rrbracket \otimes \llbracket \underline{\Gamma}_1 \rrbracket) \otimes (\llbracket \underline{\Gamma}'_1 \rrbracket \otimes \llbracket \underline{\Gamma}'_2 \rrbracket) \arrow[r, "\alpha"] \arrow[d, "(g_2 \otimes g_1) \otimes (\llbracket \underline{\mathtt{M}}'_1 \rrbracket \otimes \llbracket \underline{\mathtt{M}}'_2 \rrbracket)"'] & ((\llbracket \underline{\Gamma}_2 \rrbracket \otimes \llbracket \underline{\Gamma}_1 \rrbracket) \otimes \llbracket \underline{\Gamma}'_1 \rrbracket) \otimes \llbracket \underline{\Gamma}'_2 \rrbracket \arrow[d, "((g_2 \otimes g_1) \otimes \llbracket \underline{\mathtt{M}}'_1 \rrbracket) \otimes \llbracket \underline{\mathtt{M}}'_2 \rrbracket"] \\
(\mathcal{O}\llbracket \vec{\eta}_2 \rrbracket \otimes \mathcal{O}\llbracket \vec{\eta}_1 \rrbracket) \otimes (\mathcal{J}_{\mathrm{cd}}\llbracket \Xi_1 \rrbracket \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_2 \rrbracket) \arrow[r, "\alpha"] \arrow[d, "\theta_2 \otimes \mathrm{id}"'] & ((\mathcal{O}\llbracket \vec{\eta}_2 \rrbracket \otimes \mathcal{O}\llbracket \vec{\eta}_1 \rrbracket) \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_1 \rrbracket) \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_2 \rrbracket \arrow[d, "(\theta_2 \otimes \mathrm{id}) \otimes \mathrm{id}"] \\
(\mathcal{O}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \vec{\eta}_1 \rrbracket) \otimes (\mathcal{J}_{\mathrm{cd}}\llbracket \Xi_1 \rrbracket \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_2 \rrbracket) \arrow[r, "\alpha"] \arrow[d, equal] & (\mathcal{O}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \vec{\eta}_1 \rrbracket) \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_1 \rrbracket) \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_2 \rrbracket \arrow[dd, "\vartheta_{\llbracket \Xi_1 \rrbracket}(f') \otimes \mathrm{id}"] \\
\mathcal{O}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \vec{\eta}_1 \rrbracket) \otimes \mathcal{J}_{\mathrm{cd}}(\llbracket \Xi_1 \rrbracket \otimes \llbracket \Xi_2 \rrbracket) \arrow[d, "\vartheta_{\llbracket \Xi_1 \rrbracket \otimes \llbracket \Xi_2 \rrbracket}(f)"'] & \\
\mathcal{O}\llbracket \eta \rrbracket & \mathcal{O}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \eta' \rrbracket) \otimes \mathcal{J}_{\mathrm{cd}}\llbracket \Xi_2 \rrbracket \arrow[l, "\vartheta_{\llbracket \Xi_2 \rrbracket}\llbracket \mathtt{M} \rrbracket"]
\end{tikzcd}
$$

(the definitions of the morphisms g_1, g_2, f, and f' are provided in figure 6.3)

Figure 6.2: An instance of soundness 1

Shorthand notations

$$\begin{tikzcd}
\llbracket \Gamma_{j1} \rrbracket \otimes \cdots \otimes \llbracket \Gamma_{jn_j} \rrbracket = \llbracket \underline{\Gamma}_j \rrbracket \arrow[r, "g_j"] \arrow[d, equal] \arrow[dr, phantom, "=_{def}"] & \mathcal{O}\llbracket \vec{\eta}_j \rrbracket = \mathcal{O}(\llbracket \eta_{j1} \rrbracket \otimes \cdots \otimes \llbracket \eta_{jn_j} \rrbracket) \\
\llbracket \Gamma_{j1} \rrbracket \otimes \cdots \otimes \llbracket \Gamma_{jn_j} \rrbracket \arrow[r, "{\llbracket \underline{\mathtt{M}}_j \rrbracket = \llbracket \mathtt{M}_{j1} \rrbracket \otimes \cdots \otimes \llbracket \mathtt{M}_{jn_j} \rrbracket}"'] & \mathcal{O}\llbracket \eta_{j1} \rrbracket \otimes \cdots \otimes \mathcal{O}\llbracket \eta_{jn_j} \rrbracket \arrow[u, "\theta_{n_j}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{J}_{\mathsf{ac}}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \vec{\eta}_1 \rrbracket) \otimes (\llbracket \Xi_1 \rrbracket \otimes \llbracket \Xi_2 \rrbracket) \arrow[r, "\mathsf{f}"] \arrow[d, equal] & \mathcal{J}_{\mathsf{ac}}\llbracket \eta \rrbracket \\
(\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_2 \rrbracket \otimes \mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_1 \rrbracket) \otimes (\llbracket \Xi_1 \rrbracket \otimes \llbracket \Xi_2 \rrbracket) \arrow[dd, "\kappa"'] \arrow[dr, phantom, "=_{def}"] & \\
& \mathcal{J}_{\mathsf{ac}}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \eta' \rrbracket) \otimes \llbracket \Xi_2 \rrbracket \arrow[uu, "\llbracket \mathtt{M} \rrbracket"'] \\
(\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_2 \rrbracket \otimes (\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_1 \rrbracket \otimes \llbracket \Xi_1 \rrbracket)) \otimes \llbracket \Xi_2 \rrbracket \arrow[r, "(\mathrm{id} \otimes \llbracket \mathtt{M}' \rrbracket) \otimes \mathrm{id}"'] & (\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_2 \rrbracket \otimes \mathcal{J}_{\mathsf{ac}}\llbracket \eta' \rrbracket) \otimes \llbracket \Xi_2 \rrbracket \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{J}_{\mathsf{ac}}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \vec{\eta}_1 \rrbracket) \otimes \llbracket \Xi_1 \rrbracket \arrow[r, "\mathsf{f}'"] \arrow[d, equal] & \mathcal{J}_{\mathsf{ac}}(\llbracket \vec{\eta}_2 \rrbracket \otimes \llbracket \eta' \rrbracket) \arrow[dd, equal] \\
(\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_2 \rrbracket \otimes \mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_1 \rrbracket) \otimes \llbracket \Xi_1 \rrbracket \arrow[d, "\bar{\alpha}"'] \arrow[r, phantom, "=_{def}"] & \\
\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_2 \rrbracket \otimes (\mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_1 \rrbracket \otimes \llbracket \Xi_1 \rrbracket) \arrow[r, "\mathrm{id} \otimes \llbracket \mathtt{M}' \rrbracket"'] & \mathcal{J}_{\mathsf{ac}}\llbracket \vec{\eta}_2 \rrbracket \otimes \mathcal{J}_{\mathsf{ac}}\llbracket \eta' \rrbracket
\end{tikzcd}$$

Figure 6.3: An instance of soundness 2

and

$$\dfrac{\overline{\eta \circ \xi' \vdash_{c} \eta \circ \xi'} \qquad \dfrac{\overset{\pi}{\xi' \vdash_{c} \eta \supset \xi} \qquad \overline{\eta \vdash_{c} \eta}}{\eta, \xi' \vdash_{c} \xi}\,{\supset}\mathcal{E}}{\eta \circ \xi' \vdash_{c} \xi}\,{\circ}\mathcal{E}$$

The functor $\underline{\mathcal{O}}\colon \underline{\mathbb{A}} \longrightarrow \underline{\mathbb{D}}$ maps a formula η to the formula η°, and a derivation π of $\eta_1 \vdash_{a} \eta_2$ to the derivation

$$\dfrac{\overline{\eta_1^{\circ} \vdash_{d} \eta_1^{\circ}} \qquad \dfrac{\overline{\eta_1 \vdash_{c} \eta_1} \qquad \overset{\pi}{\eta_1 \vdash_{a} \eta_2}}{\eta_1 \vdash_{c} \eta_2}\,(\mathsf{a/c})}{\eta_1^{\circ} \vdash_{d} \eta_2^{\circ}}\,(-)^{\circ}$$

of $\eta_1^{\circ} \vdash_{d} \eta_2^{\circ}$. That it is relevant follows from the derivation of $\eta_1^{\circ} \circ \eta_2^{\circ} \vdash_{d} (\eta_1 \circ \eta_2)^{\circ}$ below and an analogous one for $\epsilon \vdash_{d} \epsilon^{\circ}$.

$$\dfrac{\overline{\eta_1^{\circ} \circ \eta_2^{\circ} \vdash_{d} \eta_1^{\circ} \circ \eta_2^{\circ}} \qquad \dfrac{\overline{\eta_1^{\circ} \vdash_{d} \eta_1^{\circ}} \qquad \overline{\eta_2^{\circ} \vdash_{d} \eta_2^{\circ}} \qquad \dfrac{\overline{\eta_1 \vdash_{d} \eta_1} \qquad \overline{\eta_2 \vdash_{d} \eta_2}}{\eta_1, \eta_2 \vdash_{d} \eta_1 \circ \eta_2}\,{\circ}\mathcal{I}}{\eta_1^{\circ}, \eta_2^{\circ} \vdash_{d} (\eta_1 \circ \eta_2)^{\circ}}\,(-)^{\circ}}{\eta_1^{\circ} \circ \eta_2^{\circ} \vdash_{d} (\eta_1 \circ \eta_2)^{\circ}}\,{\circ}\mathcal{E}$$

Thus, we have obtained a causal-deontic frame. $\underline{\vartheta}$ maps a formula η to the formula η°, and a derivation

$$\dfrac{\overline{\eta_1 \circ \xi \vdash_{c} \eta_1 \circ \xi} \qquad \overset{\pi}{\eta_1, \xi \vdash_{c} \eta_2}}{\eta_1 \circ \xi \vdash_{c} \eta_2}\,{\circ}\mathcal{E}$$

of $\eta_1 \circ \xi \vdash_{c} \eta_2$ to the derivation

$$\dfrac{\overline{\eta_1^{\circ} \circ \xi \vdash_{d} \eta_1^{\circ} \circ \xi} \qquad \dfrac{\overline{\eta_1^{\circ} \vdash_{d} \eta_1^{\circ}} \qquad \overline{\xi \vdash_{d} \xi} \qquad \overset{\pi}{\eta_1, \xi \vdash_{c} \eta_2}}{\eta_1^{\circ}, \xi \vdash_{d} \eta_2^{\circ}}\,(-)^{\circ}}{\eta_1^{\circ} \circ \xi \vdash_{d} \eta_2^{\circ}}\,{\circ}\mathcal{E}$$

of $\eta_1^{\circ} \circ \xi \vdash_{d} \eta_2^{\circ}$. That the causal-deontic transformation $\underline{\vartheta}$ satisfies the required properties is a consequence of the reductions below: they show that the uppermost derivation belongs to the same equivalence class as the lowermost derivation.

1. $\underline{\vartheta}$ is natural:

$$\dfrac{\dfrac{\overline{\xi \vdash_{d} \xi} \qquad \overset{\pi'}{\xi \vdash_{c} \xi'}}{\xi \vdash_{d} \xi'}\,(\mathsf{c/d}) \qquad \dfrac{\overline{\eta_1^{\circ} \vdash_{d} \eta_1^{\circ}} \qquad \overline{\xi' \vdash_{d} \xi'} \qquad \overset{\pi}{\eta_1, \xi' \vdash_{c} \eta_2}}{\eta_1^{\circ}, \xi' \vdash_{d} \eta_2^{\circ}}\,(-)^{\circ}}{\eta_1^{\circ}, \xi \vdash_{d} \eta_2^{\circ}}\,\mathsf{sub}$$

reduces to

$$\dfrac{\overline{\eta_1^\circ \vdash_{\mathsf{d}} \eta_1^\circ} \quad \dfrac{\overline{\xi \vdash_{\mathsf{d}} \xi} \quad \begin{array}{c}\pi'\\ \xi \vdash_{\mathsf{c}} \xi'\end{array}}{\xi \vdash_{\mathsf{d}} \xi'}\,(\mathsf{c}/\mathsf{d}) \quad \begin{array}{c}\pi\\ \eta_1,\xi' \vdash_{\mathsf{c}} \eta_2\end{array}}{\eta_1^\circ,\xi \vdash_{\mathsf{d}} \eta_2^\circ}\,(-)^\circ$$

β-reduces to

$$\dfrac{\overline{\eta_1^\circ \vdash_{\mathsf{d}} \eta_1^\circ} \quad \overline{\xi \vdash_{\mathsf{d}} \xi} \quad \dfrac{\begin{array}{c}\pi'\\ \xi \vdash_{\mathsf{c}} \xi'\end{array} \quad \begin{array}{c}\pi\\ \eta_1,\xi' \vdash_{\mathsf{c}} \eta_2\end{array}}{\eta_1,\xi \vdash_{\mathsf{c}} \eta_2}\,\mathsf{sub}}{\eta_1^\circ,\xi_1 \vdash_{\mathsf{d}} \eta_2^\circ}\,(-)^\circ$$

2. Coherence for $\underline{\vartheta}$:

$$\dfrac{\dfrac{\overline{\eta_1^\circ \vdash_{\mathsf{d}} \eta_1^\circ} \quad \overline{\xi_1 \vdash_{\mathsf{d}} \xi_1} \quad \begin{array}{c}\pi'\\ \eta_1,\xi_1 \vdash_{\mathsf{c}} \eta'\end{array}}{\eta_1^\circ,\xi_1 \vdash_{\mathsf{d}} (\eta')^\circ}\,(-)^\circ \quad \dfrac{\overline{\eta_2^\circ \vdash_{\mathsf{d}} \eta_2^\circ} \quad \overline{(\eta')^\circ \vdash_{\mathsf{d}} (\eta')^\circ} \quad \overline{\xi_2 \vdash_{\mathsf{d}} \xi_2} \quad \begin{array}{c}\pi\\ \eta_2,\eta',\xi_2 \vdash_{\mathsf{c}} \eta\end{array}}{\eta_2^\circ,(\eta')^\circ,\xi_2 \vdash_{\mathsf{d}} \eta^\circ}\,(-)^\circ}{(\eta_2^\circ,\eta_1^\circ),(\xi_1,\xi_2) \vdash_{\mathsf{d}} \eta^\circ}\,\mathsf{sub}$$

reduces to

$$\dfrac{\overline{\eta_2^\circ \vdash_{\mathsf{d}} \eta_2^\circ} \quad \dfrac{\overline{\eta_1^\circ \vdash_{\mathsf{d}} \eta_1^\circ} \quad \overline{\xi_1 \vdash_{\mathsf{d}} \xi_1} \quad \begin{array}{c}\pi'\\ \eta_1,\xi_1 \vdash_{\mathsf{c}} \eta'\end{array}}{\eta_1^\circ,\xi_1 \vdash_{\mathsf{d}} (\eta')^\circ}\,(-)^\circ \quad \overline{\xi_2 \vdash_{\mathsf{d}} \xi_2} \quad \begin{array}{c}\pi\\ \eta_2,\eta',\xi_2 \vdash_{\mathsf{c}} \eta\end{array}}{(\eta_2^\circ,\eta_1^\circ),(\xi_1,\xi_2) \vdash_{\mathsf{d}} \eta^\circ}\,(-)^\circ$$

β-reduces to

$$\dfrac{\overline{\eta_2^\circ \vdash_{\mathsf{d}} \eta_2^\circ} \quad \overline{\eta_1^\circ \vdash_{\mathsf{d}} \eta_1^\circ} \quad \overline{\xi_1 \vdash_{\mathsf{d}} \xi_1} \quad \overline{\xi_2 \vdash_{\mathsf{d}} \xi_2} \quad \dfrac{\begin{array}{c}\pi'\\ \eta_1,\xi_1 \vdash_{\mathsf{c}} \eta'\end{array} \quad \begin{array}{c}\pi\\ \eta_2,\eta',\xi_2 \vdash_{\mathsf{c}} \eta\end{array}}{(\eta_2,\eta_1),(\xi_1,\xi_2) \vdash_{\mathsf{c}} \eta}\,\mathsf{sub}}{(\eta_2^\circ,\eta_1^\circ),(\xi_1,\xi_2) \vdash_{\mathsf{d}} \eta^\circ}\,(-)^\circ$$

□

6.3 Algebraic models as instances

We complete our investigation of a semantics of proofs with a class of sound and complete algebraic models (see definition 6.3.5 and proposition 6.3.7) for the basic causal-deontic system. To get straightforward proofs of soundness and completeness we follow the strategy adopted in [POY04], i.e. we introduce an intermediate calculus in figure 6.4 and show that it is equivalent to the sequent calculus system of chapter 3 (see lemma 6.3.3). Note that the algebraic models

$$\frac{\xi_1 \circ \eta \vdash_c \xi_2}{\xi_1 \vdash_c \eta \Supset \xi_2} \qquad (\eta \Supset \xi) \circ \eta \vdash_c \xi$$

$$\frac{\eta_1 \vdash_a \eta_2}{\eta_1 \vdash_c \eta_2} \qquad \frac{\xi_1 \vdash_c \xi_2}{\xi_1 \vdash_d \xi_2}$$

$$\epsilon \vdash_d \epsilon^\circ \qquad \eta_1^\circ \circ \eta_2^\circ \vdash_d (\eta_1 \circ \eta_2)^\circ \qquad \frac{\eta_1 \circ \xi \vdash_c \eta_2}{\eta_1^\circ \circ \xi \vdash_d \eta_2^\circ}$$

$$\gamma \vdash_x \gamma \qquad \gamma \vdash_x \gamma \circ \gamma \qquad \gamma_1 \circ \gamma_2 \vdash_x \gamma_2 \circ \gamma_1$$

$$\gamma_1 \circ (\gamma_2 \circ \gamma_3) \vdash_x (\gamma_1 \circ \gamma_2) \circ \gamma_3 \qquad (\gamma_1 \circ \gamma_2) \circ \gamma_3 \vdash_x \gamma_1 \circ (\gamma_2 \circ \gamma_3)$$

$$\gamma \vdash_x \gamma \circ \epsilon \qquad \gamma \circ \epsilon \vdash_x \gamma \qquad \gamma \vdash_x \epsilon \circ \gamma \qquad \epsilon \circ \gamma \vdash_x \gamma$$

$$\frac{\gamma_1 \vdash_x \gamma_1' \quad \gamma_2 \vdash_x \gamma_2'}{\gamma_1 \circ \gamma_2 \vdash_x \gamma_1' \circ \gamma_2'} \qquad \frac{\gamma_1 \vdash_x \gamma_2 \quad \gamma_2 \vdash_x \gamma_3}{\gamma_1 \vdash_x \gamma_3}$$

Figure 6.4: Lambek-style calculus

are obtained by taking the posetal collapse of the above categorical semantics. The main reason for taking into account such a degenerate instance of the semantics of proof is that algebraic models provide means to illustrate how the Kripke semantics of chapter 4 fits into the general semantic framework. This is done by showing that Kripke models induce algebraic models (see proposition 6.3.8). Note however that our characterisation is not entirely algebraic: we take advantage of the forcing relation when defining the causal-deontic algebra.

Remark 6.3.1 In [POY04], a calculus very similar to the one of figure 6.4 is called a Hilbert system. However, since the calculus should rather be seen as a deductive system in the sense of [LS86] than as a Hilbert system of axioms and rules, we call it Lambek-style calculus. Note that the calculus of figure 6.4 can be regarded as the set of production rules for the morphisms associated to a causal-deontic category given in definition 6.2.1.

Lemma 6.3.2 $\eta_1^\circ \circ \cdots \circ \eta_n^\circ \vdash_{\mathrm{d}} (\eta_1 \circ \cdots \circ \eta_n)^\circ$.

Proof By the associativity of the relevant conjunction we may assume that the bracketing on the lefthand side and the righthand side of $\vdash_{\mathrm{d}}$ is of the form $(\cdots((a_1 \circ a_2) \circ a_2) \circ \cdots) \circ a_n$. The proof proceeds then by induction on n. □

Lemma 6.3.3 $\gamma_1 \Rightarrow_{\mathrm{x}} \gamma_2$ if and only if $\gamma_1 \vdash_{\mathrm{x}} \gamma_2$.

Proof The difficult part is to show that $\gamma_1 \Rightarrow_{\mathrm{x}} \gamma_2$ implies $\gamma_1 \vdash_{\mathrm{x}} \gamma_2$. We consider only the case where the sequent calculus derivation ends with an instance of the $(-)^\circ$ rule. Note that this rule may be rendered as follows:

$$\frac{\eta_1 \circ \cdots \circ \eta_n \circ \xi \Rightarrow_{\mathrm{c}} \eta}{\eta_1^\circ \circ \cdots \circ \eta_n^\circ \circ \xi \Rightarrow_{\mathrm{d}} \eta^\circ}\,(-)^\circ$$

That the entailment $\eta_1^\circ \circ \cdots \circ \eta_n^\circ \circ \xi \vdash_{\mathrm{d}} \eta^\circ$ is derivable in the Lambek-style system is then shown as follows:

(1)	$\xi \vdash_{\mathrm{d}} \xi$	
(2)	$\eta_1^\circ \circ \cdots \circ \eta_n^\circ \vdash_{\mathrm{d}} (\eta_1 \circ \cdots \circ \eta_n)^\circ$	
(3)	$\eta_1^\circ \circ \cdots \circ \eta_n^\circ \circ \xi \vdash_{\mathrm{d}} (\eta_1 \circ \cdots \circ \eta_n)^\circ \circ \xi$	(1) and (2)
(4)	$\eta_1 \circ \cdots \circ \eta_n \circ \xi \vdash_{\mathrm{d}} \eta$	hypothesis
(5)	$(\eta_1 \circ \cdots \circ \eta_n)^\circ \circ \xi \vdash_{\mathrm{d}} \eta^\circ$	(4)
(6)	$\eta_1^\circ \circ \cdots \circ \eta_n^\circ \circ \xi \vdash_{\mathrm{d}} \eta^\circ$	(3) and (5)

□

Definition 6.3.4 A *causal-deontic algebra* (A, C, D, i, j, o) consists of three relevant monoids

$$A = (A, \cdot, 1, \leq_{\mathrm{a}}) \qquad C = (C, \cdot, 1, \leq_{\mathrm{c}}) \qquad D = (D, \cdot, 1, \leq_{\mathrm{d}})$$

together with three monotone functions

$$i : A \longrightarrow C \qquad j : C \longrightarrow D \qquad o : A \longrightarrow D$$

such that the following conditions are satisfied:

1. i and j are monoid homomorphisms;
2. C is residuated (closed) with respect to $i(A)$;[2]
3. o is such that $1 \leq_{\mathrm{d}} o(1)$ and $o(a_1) \cdot o(a_2) \leq_{\mathrm{d}} o(a_1 \cdot a_2)$;

[2] This means that there is a binary operation $\multimap$ such that, for all $a \in A$ and $c_1, c_2 \in C$, $i(a) \cdot c_1 \leq_{\mathrm{c}} c_2$ if and only if $c_1 \leq_{\mathrm{c}} i(a) \multimap c_2$.

4. if $i(a_1) \cdot c \leq_c i(a_2)$ then $o(a_1) \cdot j(c) \leq_d o(a_2)$.

Definition 6.3.5 An *algebraic model* $(A, C, D, i, j, o, [-])$ consists of a causal-deontic algebra (A, C, D, i, j, o) together with an interpretation function $[-]$ that maps assertive, pure causal, and causal-deontic formulae to elements of A, C, and D, respectively:

1. assertive formulae: $[\eta_p] = a \in A$, $[\epsilon] = 1$, and $[\eta_1 \circ \eta_2] = [\eta_1] \cdot [\eta_2]$;

2. pure causal formulae: $[\eta] = i([\eta])$, $[\eta \Supset \xi] = i([\eta]) \multimap [\xi]$, and $[\xi_1 \circ \xi_2] = [\xi_1] \cdot [\xi_2]$;

3. causal-deontic formulae: $[\eta^\circ] = o([\eta])$, $[\xi] = j([\xi])$, and $[\gamma_1 \circ \gamma_2] = [\gamma_1] \cdot [\gamma_2]$.

Remark 6.3.6 Following the conventions of remark 6.2.3, the interpretation $[\eta]$ of an assertive formula η in C is obtained as the image along the function i of the interpretation of η in A; the interpretation $[\xi]$ of a pure causal formula ξ in D follows a similar pattern.

Proposition 6.3.7 (soundness & completeness) $\gamma_1 \vdash_x \gamma_2$ if and only if $[\gamma_1] \leq_x [\gamma_2]$ holds for all interpretations in all causal-deontic algebras.

Proof We omit the straightforward proof. □

Proposition 6.3.8 (relation between models) Given a basic Kripke model $(W, U, \lhd, \Vdash)$, we can construct an algebraic model $(A, C, D, i, j, o, [-])$.

Proof Let the elements of D be the downward closed subsets of W. Given downward closed subsets W_1 and W_2 of W, if we let $W_1 \cdot W_2 = \{w \in W \mid w \preceq w_1 \cdot w_2 \text{ for } w_1 \in W_1, w_2 \in W_2\}$ and $1 = \{w \in W \mid w \preceq 1\}$ then $D = (D, \cdot, 1, \subseteq)$ is a relevant monoid where $\subseteq$ stands for subset inclusion. If A is the smallest set of downward closed subsets X of D such that, for all $x \in X$, $x \Vdash \eta$ for any assertive formula η and C the smallest set of downward closed subsets Y of D such that, for all $y \in Y$, $y \Vdash \xi$ for any pure causal formula ξ, then $A = (A, \cdot, 1, \subseteq)$ and $C = (C, \cdot, 1, \subseteq)$ are relevant monoids. In this setting i and j are given by identities; that C is closed with respect to $A = i(A)$ follows immediately from the definition of forcing. The function $o \colon A \longrightarrow D$ associates to a downward closed subset X of A the downward closed subset $\{w \in W \mid u \lhd w \text{ for } u \in X\}$ of D; that it satisfies the required properties follows from the definition of Kripke model. Thus we have a causal-deontic algebra (A, C, D, i, j, o). The algebraic model is then obtained by setting $[\gamma] = \{w \in W \mid w \Vdash \gamma\}$. □

6.4 Discussion and final remarks

We thus conclude our investigation of the proof theory and semantics of the basic causal-deontic system. In summary, we have considered sequent calculus and natural deduction presentations for it, presented a term assignment for natural deduction, and provided classes of sound and complete Kripke, algebraic, and categorical models. Nonetheless, there are some issue that should be addressed in future work. First, there is room for improvement with respect to the relation between Kripke and algebraic models. On the one hand, we have already pointed out at the beginning of the previous section that the construction of the causal-deontic algebra in the proof of proposition 6.3.8 is not entirely algebraic. On the other hand, it would be interesting to investigate whether Kripke models are induced by algebraic models, thus providing means to get some sort of representation result. Second, we conjecture that the type theory proposed in the previous chapter provides the internal language (see for instance [MMdPR05] and [Mai08]) for causal-deontic categories. Third, as we have argued in [Ran08b], there are natural deduction presentations of the basic intuitionistic modal logic **K** that bear a close relationship to fibrations and hence to split indexed categories. It thus seems that the above categorical construction can be considered as a particular instance of some general scheme.

Chapter 7

Extended deontic system

We consider now the logical system corresponding to the extended causal-deontic fragment $\mathcal{L}_E^P$ provided in definition 2.3.1. Remember that it simply corresponds to the basic causal-deontic fragment $\mathcal{L}_B^P$ of the pragmatic language with normative formulae η° defined by the grammar $\eta_p^\circ \mid \eta^\circ /\!/ \eta \mid \eta^\circ \circ \eta^\circ$ where p is an element of $\mathcal{Q}$, the set of atomic propositions that satisfy the content condition. We shall discuss only how certain definitions have to be rearranged for the new system. We conjecture however that, by adapting the proofs provided in the previous chapters, one can actually get the expected soundness and completeness results.

7.1 New rules and Kripke models

The extended causal-deontic system is obtained by adding the rules of figure 7.1 to the basic causal-deontic rules of figure 3.1 without the $(-)^\circ$ rule. The main aim of this section is to show how the Kripke models of chapter 4 can be adapted to the extended system.

Remark 7.1.1 The new version of the $(-)^\circ$ rule is based on the restriction that $\vec{\eta} \neq \epsilon$. This condition serves the purpose to guarantee that Hume's law cannot be violated. Note that by the definition of the grammar for η° we also have that $rad(\vec{\eta}, \eta) \subseteq \mathcal{Q}$. As a consequence of the `dis` axiom, we have that both $\eta_1^\circ \circ \eta_2^\circ$ and $(\eta_1 \circ \eta_2)^\circ$ are interderivable.

Remark 7.1.2 Both the left and the right rule for $/\!/$ emphasise that the connective can be considered as a proper conditional implication. As in remark 3.1.4 we have that the extended deontic system is closely related to Horn fragments of multiplicative linear logic.

$$\frac{}{(\eta_1 \circ \eta_2)^\circ \Rightarrow_{\mathrm{d}} \eta_1^\circ \circ \eta_2^\circ}\ \mathtt{dis} \qquad \frac{\vec{\eta}, \Xi \Rightarrow_{\mathrm{c}} \eta}{(\vec{\eta})^\circ, \Xi \Rightarrow_{\mathrm{d}} \eta^\circ}\ (-)^\circ\ [\vec{\eta} \neq \epsilon]$$

$$\frac{\Gamma, \eta_1 \Rightarrow_{\mathrm{d}} \eta_2^\circ}{\Gamma \Rightarrow_{\mathrm{d}} \eta_2^\circ /\!\!/ \eta_1}\ \mathcal{R}/\!\!/ \qquad \frac{\Gamma_1 \Rightarrow_{\mathrm{d}} \eta_1 \quad \eta_2^\circ, \Gamma_2 \Rightarrow_{\mathrm{d}} \gamma}{\eta_2^\circ /\!\!/ \eta_1, \Gamma_1, \Gamma_2 \Rightarrow_{\mathrm{d}} \gamma}\ \mathcal{L}/\!\!/$$

Figure 7.1: Extended deontic rules

Definition 7.1.3 A *preordered semigroup* $(S, \cdot, \leq)$ is a commutative semigroup $(S, \cdot, 1)$ endowed with a preorder $\leq$ such that, for all $s_1, s_1', s_2, s_2' \in S$, if $s_1 \leq s_1'$ and $s_2 \leq s_2'$ then $s_1 \cdot s_2 \leq s_1' \cdot s_2'$.

Definition 7.1.4 A preordered semigroup $(S, \cdot, \leq)$ is called *relevant* if, for all $s \in S$, $s \leq s \cdot s$. For the sake of conciseness we shall use relevant semigroup instead of relevant preordered semigroup throughout this section.

Definition 7.1.5 Let $W = (W, \cdot, 1, \preceq)$ be a relevant monoid and $V = (V, \cdot, \preceq)$ be a relevant subsemigroup of W. An *extended Kripke frame* is a triple $(W, V, \lhd)$ where $\lhd \subseteq V \times W$ is a binary relation such that

1. for all $v \in V$ and $w, w', w'' \in W$, if $v \lhd w$ and $w \preceq w' \cdot w''$ then there exist $v', v'' \in V$ such that $v' \lhd w'$, $v'' \lhd w''$, and $v \preceq v' \cdot v''$;
2. for all $v, v', v'' \in V$ and $w \in W$, if $v \lhd w$ and $v \preceq v' \cdot v''$ then there exist $w', w'' \in W$ such that $v' \lhd w'$, $v'' \lhd w''$, and $w \preceq w' \cdot w''$;
3. for all $v' \in V$ and $w, w', w'' \in W$, if $v' \lhd w'$ and $w \preceq w' \cdot w''$ then, for all $v \in V$, if $v \lhd w$ then $v \preceq v' \cdot w''$.

Definition 7.1.6 An *extended Kripke model* $(W, V, \lhd, \Vdash)$ is an extended Kripke frame $(W, V, \lhd)$ endowed with a forcing relation $\Vdash \subseteq W \times \Gamma$. Given a downward closed subset $\eta_p^\star$ of W for each propositional atom p, the forcing relation is defined as follows:

1. $w \Vdash \eta_p$ iff $w \in \eta_p^\star$;
2. $w \Vdash \epsilon$ iff $w \preceq 1$;

3. $w \Vdash \eta^\circ$ iff for all $v \in V$ such that $v \lhd w$, $v \Vdash \eta$;

4. $w \Vdash \eta_2^\circ /\!\!/ \eta_1$ iff for all $w' \in W$ such that $w' \Vdash \eta_1$, $w \cdot w' \Vdash \eta_2^\circ$;

5. $w \Vdash \eta \supset \xi$ iff for all $w' \in W$ such that $w' \Vdash \eta$, $w \cdot w' \Vdash \xi$;

6. $w \Vdash \gamma_1 \circ \gamma_2$ iff there exist $w_1, w_2 \in W$ such that $w \preceq w_1 \cdot w_2$, $w_1 \Vdash \gamma_1$, and $w_2 \Vdash \gamma_2$.

Remark 7.1.7 It is worth mentioning the main differences between basic and extended Kripke models. First, the set V contains only the possible worlds that force an assertive formula η where all radicals are in $\mathcal{Q}$ and therefore there might be some $w \notin V$ that forces an assertive formula η. Note however that this is the case only if $rad(\eta) \not\subseteq \mathcal{Q}$. Second, conditions (1) and (2) of definition 7.1.5 make sure that, for all possible worlds $w \in W$, w forces the normative formula $\eta_1^\circ \circ \eta_2^\circ$ if and only if w forces the normative formula $(\eta_1 \circ \eta_2)^\circ$, thus guaranteeing that the $(-)^\circ$ operator preserves relevant conjunctions in a strong sense. Condition (3) of definition 7.1.5 remains unchanged with respect to condition (3) of definition 4.1.3.

7.2 Towards a semantics of proofs

We briefly sketch how the the semantics of proofs provided in chapter 6 can be adapted to deal with the extended causal-deontic system. The key observation is that we need to provide a refined version of causal-deontic frame, all other definitions requiring only minor adjustments.

Remark 7.2.1 Since in the new version of the $(-)^\circ$ rule we have the syntactic restriction that $\vec{\eta} \neq \epsilon$, we need to take care of this aspect also in the categorical semantics. The solution we propose is based on the concept of semi-monoidal or unit-free monoidal category (see for instance [HHS05]): roughly speaking, it is the categorical counterpart of the concept of semigroup. A unit-free monoidal category is just an ordinary monoidal category where the unit object I and the canonical isomorphisms λ and ρ are missing. Since we have the structural rule of exchange we shall assume without further mention that unit-free monoidal categories are symmetric. Similarly, a unit-free relevant category is just a relevant category where I, λ, and ρ are missing. Unit-free relevant functors are defined as the obvious structure preserving functors.

Definition 7.2.2 A *refined causal-deontic frame* $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathbb{A}^*, \mathbb{D}^*, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{O})$ consists of three relevant categories

$$\mathbb{A} = (\mathbb{A}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta) \qquad \mathbb{C} = (\mathbb{C}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta) \qquad \mathbb{D} = (\mathbb{D}, \otimes, \mathrm{I}, \alpha, \lambda, \rho, \tau, \delta)$$

and two unit-free relevant categories

$$\mathbb{A}^* = (\mathbb{A}^*, \otimes, \alpha, \tau, \delta) \qquad\qquad \mathbb{D}^* = (\mathbb{D}^*, \otimes, \alpha, \tau, \delta)$$

together with two relevant functors

$$\mathcal{J}_{\mathrm{ac}} = (\mathcal{J}_{\mathrm{ac}}, \iota_2, \iota_0) \colon \mathbb{A} \longrightarrow \mathbb{C} \qquad \mathcal{J}_{\mathrm{cd}} = (\mathcal{J}_{\mathrm{cd}}, \iota_2, \iota_0) \colon \mathbb{C} \longrightarrow \mathbb{D}$$

and a unit-free relevant functor

$$\mathcal{O} = (\mathcal{O}, \theta_2, \theta_0) \colon \mathbb{A}^* \longrightarrow \mathbb{D}^*$$

such that the following conditions are satisfied:

1. $\mathbb{A}^* \subseteq_{full} \mathbb{A}$ and $\mathbb{D}^* \subseteq_{full} \mathbb{D}$;[1]
2. $\mathcal{J}_{\mathrm{ac}}$, $\mathcal{J}_{\mathrm{cd}}$, and $\mathcal{O}$ are strict functors;
3. $\mathbb{C}$ is closed with respect to its subcategory $\mathcal{J}_{\mathrm{ac}}\mathbb{A}$;
4. $\mathbb{D}^*$ is closed with respect to the category $\mathcal{J}_{\mathrm{cd}}(\mathcal{J}_{\mathrm{ac}}\mathbb{A})$.

Remark 7.2.3 One may think of $\mathbb{A}^*$ and $\mathbb{D}^*$ as the unit-free subcategories of $\mathbb{A}$ and $\mathbb{D}$ that have among their objects the interpretations of the formulae obtained by the grammars $\eta' := \eta_p \mid \eta' \circ \eta'$ and $\eta^\circ := \eta_p^\circ \mid \eta^\circ /\!\!/ \eta \mid \eta^\circ \circ \eta^\circ$ where p ranges over the set Q.

Remark 7.2.4 To complete the definition of the categorical model for the extended causal-deontic system we only need to specify how the definition of causal-deontic transformation has to be modified. It suffices to change the definition of restriction functor in such a way that it works with the unit-free relevant category $\mathbb{A}^*$ instead of the relevant category $\mathbb{A}$, i.e. to consider a functor $\mathcal{R}_{\mathbb{C}}^{\mathbb{A}^*} \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{SCat}$ whose image semi-category has objects $\mathcal{J}_{\mathrm{ac}}\mathrm{A}$ of $\mathbb{C}$ where A is an object of $\mathbb{A}^*$ instead of objects $\mathcal{J}_{\mathrm{ac}}\mathrm{A}$ of $\mathbb{C}$ where A is an object of $\mathbb{A}$. The intuition behind this choice is that the relevant functor $\mathcal{O}$ modelling the $(-)^\circ$ operator can only be applied to objects and morphisms of the unit-free relevant subcategory $\mathbb{A}^*$ of $\mathbb{A}$. We omit further details and simply point out that the definition of the causal-deontic transformation $\vartheta \colon \mathcal{R}_{\mathbb{C}}^{\mathbb{A}^*} \longrightarrow \mathcal{E}_{\mathbb{C}}^{\mathbb{D}}$ and thus the definition of causal-deontic category can then be adapted in a straightforward way.

[1] That is, $\mathbb{A}^*$ is a full subcategory of $\mathbb{A}$ and $\mathbb{D}^*$ is a full subcategory of $\mathbb{D}$.

7.3 Discussion and final remarks

In this rather short chapter he have shown how to modify certain definitions in such a way that the results of the previous four chapters should carry over to the extended causal-deontic fragment of the pragmatic language in a straightforward way. Instead of going into the details we have chosen to focus on the Kripke semantics and the semantics of proofs. The upshot is that, with only a few minor modifications, the basic causal-deontic system can be extended in such a way to support a notion of hypothetical norm or conditional obligation. It is worth mentioning that the concept of conditional obligation considered here is a rather non-standard one, in the sense that conditional obligation is usually considered as a dyadic operator (see for instance [Che80, section 10.2]). However, in the light of remarks 3.1.4 and 7.1.2, we can reduce each conditional obligation to one of the form $(\eta^{\circ}_{p_1} \circ \cdots \circ \eta^{\circ}_{p_n}) /\!\!/ (\eta_{p'_1} \circ \cdots \circ \eta_{p'_m})$ and thus avoid iterations of $/\!\!/$. We have not investigated whether it is possible to provide a pragmatic interpretation of standard conditional obligation in our framework and leave this issue for future research.

Chapter 8

Full intuitionistic system

We consider the full intuitionistic fragment $\mathcal{L}_F^P$ of the pragmatic language provided in definition 2.3.1. We shall briefly discuss its main properties and then focus on the relationship between the system considered in this thesis and the system **ILP** considered in [BD02].

8.1 On the pragmatic connectives

The sequent calculus presentation of the full intuitionistic system is obtained by adding the rules of figure 8.1 to the sequent calculus presentation of the basic causal-deontic system. Except for the $(\mathtt{d}/\mathtt{p})$ rule, the pragmatic rules are exactly the standard rules of propositional intuitionistic logic. As for the causal-deontic system, we have that the causal-deontic fragment of the calculus is included in the pragmatic fragment: this is a consequence of the $(\mathtt{d}/\mathtt{p})$ rule.

Theorem 8.1.1 (cut elimination) If a sequent $\Delta \Rightarrow_{\mathtt{x}} \delta$ is derivable according to the rules of figures 3.1 and 8.1 then it has a cut-free derivation.

Proof One can extend the proof of theorem 3.1.8 in a straightforward way. □

Remark 8.1.2 It is folklore that bicartesian closed categories are categorical models of propositional intuitionistic logic (see for instance [LS86]). Hence, we only have to introduce some categorical structure that allows us to model the $(\mathtt{d}/\mathtt{p})$ rule. Similarly to what we have already seen in previous chapters, we simply need to add a bicartesian closed category $\mathbb{P}$ to the definition of causal-deontic frame and impose that there is a relevant functor $\mathcal{J}_{\mathrm{dp}} = (\mathcal{J}_{\mathrm{dp}}, \iota_2, \iota_0) \colon \mathbb{D} \longrightarrow$

$$\frac{\Gamma \Rightarrow_{\mathrm{d}} \gamma}{\Gamma, \Delta \Rightarrow_{\mathrm{p}} \gamma}\,(\mathrm{d/p}) \qquad \frac{}{\Delta \Rightarrow_{\mathrm{p}} \mathbb{J}}\,\mathbb{J}\,\mathrm{ax} \qquad \frac{}{\mathbb{U}, \Delta \Rightarrow_{\mathrm{p}} \delta}\,\mathbb{U}\,\mathrm{ax}$$

$$\frac{\delta', \delta', \Delta \Rightarrow_{\mathrm{p}} \delta}{\delta', \Delta \Rightarrow_{\mathrm{p}} \delta}\,\mathrm{con} \qquad \frac{\Delta \Rightarrow_{\mathrm{p}} \delta' \quad \delta', \Delta \Rightarrow_{\mathrm{p}} \delta}{\Delta \Rightarrow_{\mathrm{p}} \delta}\,\mathrm{cut}$$

$$\frac{\delta_i, \Delta \Rightarrow_{\mathrm{p}} \delta}{\delta_1 \cap \delta_2, \Delta \Rightarrow_{\mathrm{p}} \delta}\,\mathcal{L}\cap \ (i=1,2) \qquad \frac{\Delta \Rightarrow_{\mathrm{p}} \delta_1 \quad \Delta \Rightarrow_{\mathrm{p}} \delta_2}{\Delta \Rightarrow_{\mathrm{p}} \delta_1 \cap \delta_2}\,\mathcal{R}\cap$$

$$\frac{\Delta \Rightarrow_{\mathrm{p}} \delta_i}{\Delta \Rightarrow_{\mathrm{p}} \delta_1 \cup \delta_2}\,\mathcal{R}\cup \ (i=1,2) \qquad \frac{\delta_1, \Delta \Rightarrow_{\mathrm{p}} \delta \quad \delta_2, \Delta \Rightarrow_{\mathrm{p}} \delta}{\delta_1 \cup \delta_2, \Delta \Rightarrow_{\mathrm{p}} \delta}\,\mathcal{L}\cup$$

$$\frac{\Delta \Rightarrow_{\mathrm{p}} \delta_1 \quad \delta_2, \Delta \Rightarrow_{\mathrm{p}} \delta}{\delta_1 \supset \delta_2, \Delta \Rightarrow_{\mathrm{p}} \delta}\,\mathcal{L}\supset \qquad \frac{\Delta, \delta_1 \Rightarrow_{\mathrm{p}} \delta_2}{\Delta \Rightarrow_{\mathrm{p}} \delta_1 \supset \delta_2}\,\mathcal{R}\supset$$

Figure 8.1: (d/p) and pragmatic rules

$\mathbb{P}$. Note that we do not require that $\mathcal{J}_{\mathrm{dp}}$ is either strict or strong, the main reason being that $\gamma_1 \cap \gamma_2 \Rightarrow_{\mathrm{p}} \gamma_1 \circ \gamma_2$ is derivable in sequent calculus whereas $\gamma_1 \circ \gamma_2 \Rightarrow_{\mathrm{p}} \gamma_1 \cap \gamma_2$ is not. To make things more formal let us spell out these ideas in the following definition.

Definition 8.1.3 A *pragmatic category* $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathbb{P}, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{J}_{\mathrm{dp}}, \mathcal{O}, \vartheta)$ consists of a causal-deontic category $(\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathcal{J}_{\mathrm{ac}}, \mathcal{J}_{\mathrm{cd}}, \mathcal{O}, \vartheta)$ endowed with a bicartesian closed category $\mathbb{P}$ and a relevant functor $\mathcal{J}_{\mathrm{dp}} = (\mathcal{J}_{\mathrm{dp}}, \iota_2, \iota_0) : \mathbb{D} \longrightarrow \mathbb{P}$.

Remark 8.1.4 Unfortunately, there are some problems with the Kripke semantics. One might be tempted to think that there is already enough structure, i.e. the preorder $\preceq$, in the definition of basic Kripke frame that enables us to provide a definition of Kripke model for the full intuitionistic system by adding the usual clauses for propositional intuitionistic logic to the definition of forcing. As it is often the case, the problem arises with the units: since each $w \in W$ forces $\mathbb{J}$ we would need to have that each $w \in W$ also forces ϵ, thus giving rise to the structural rule of weakening in the sequent calculus for the basic causal-deontic system. A quick fix to this problem could be the introduction of a new preordered set of possible worlds for the pragmatic connectives and a suitable map from the basic Kripke frame to it. However, one would loose the elegance and conciseness of the original Kripke semantics.

$$\frac{}{\eta_p; \_ \Rightarrow \eta_p}\,\mathrm{ax} \qquad\qquad \frac{}{\Gamma; \mathbb{U}, \Delta \Rightarrow \delta}\,\mathbb{U}\,\mathrm{ax}$$

$$\frac{\Gamma_1; \Delta \Rightarrow \gamma \quad \gamma, \Gamma_2; \Delta \Rightarrow \delta}{\Gamma_1, \Gamma_2; \Delta \Rightarrow \delta}\,\mathtt{cut_d} \qquad\qquad \frac{\_; \Delta \Rightarrow \delta' \quad \Gamma; \delta', \Delta \Rightarrow \delta}{\Gamma; \Delta \Rightarrow \delta}\,\mathtt{cut_p}$$

$$\frac{\gamma, \gamma, \Gamma; \Delta \Rightarrow \delta}{\gamma, \Gamma; \Delta \Rightarrow \delta}\,\mathtt{con_d} \qquad\qquad \frac{\Gamma; \delta', \delta', \Delta \Rightarrow \delta}{\Gamma; \delta', \Delta \Rightarrow \delta}\,\mathtt{con_p}$$

$$\frac{\Gamma, \gamma; \Delta \Rightarrow \delta}{\Gamma; \gamma, \Delta' \Rightarrow \delta}\,\mathtt{perm}\ (\Delta \subseteq \Delta') \qquad\qquad \frac{\eta_{p_1}, \ldots, \eta_{p_n}, \Gamma; \_ \Rightarrow \eta_p}{\eta^{\circ}_{p_1}, \ldots, \eta^{\circ}_{p_n}, \Gamma; \_ \Rightarrow \eta^{\circ}_p}\,\Supset/\multimap$$

$$\frac{\Gamma_1; \_ \Rightarrow \eta_p \quad \xi, \Gamma_2; \Delta \Rightarrow \delta}{\eta_p \Supset \xi, \Gamma_1, \Gamma_2; \Delta \Rightarrow \delta}\,\mathcal{L}\Supset \qquad\qquad \frac{\Gamma, \eta_p; \_ \Rightarrow \xi}{\Gamma; \_ \Rightarrow \eta_p \Supset \xi}\,\mathcal{R}\Supset$$

$$\frac{\Gamma; \delta_i, \Delta \Rightarrow \delta}{\Gamma; \delta_1 \cap \delta_2, \Delta \Rightarrow \delta}\,\mathcal{L}\cap\ (i = 1, 2) \qquad\qquad \frac{\Gamma; \Delta \Rightarrow \delta_1 \quad \Gamma; \Delta \Rightarrow \delta_2}{\Gamma; \Delta \Rightarrow \delta_1 \cap \delta_2}\,\mathcal{R}\cap$$

$$\frac{\Delta \Rightarrow \delta_i}{\Delta \Rightarrow \delta_1 \cup \delta_2}\,\mathcal{R}\cup\ (i = 1, 2) \qquad\qquad \frac{\delta_1, \Delta \Rightarrow \delta \quad \delta_2, \Delta \Rightarrow \delta}{\delta_1 \cup \delta_2, \Delta \Rightarrow \delta}\,\mathcal{L}\cup$$

$$\frac{\_; \Delta \Rightarrow \delta_1 \quad \Gamma; \delta_2, \Delta \Rightarrow \delta}{\Gamma; \delta_1 \supset \delta_2, \Delta \Rightarrow \delta}\,\mathcal{L}\supset \qquad\qquad \frac{\Gamma; \Delta, \delta_1 \Rightarrow \delta_2}{\Gamma; \Delta \Rightarrow \delta_1 \supset \delta_2}\,\mathcal{R}\supset$$

Figure 8.2: The original rules of **ILP**

8.2 Relation to the original system

We show that the sequent calculus system given in figures 3.1 and 8.1 is a conservative extension of the system **ILP** originally presented in [BD02]. The rules of **ILP** are provided in figure 8.2.

Theorem 8.2.1 (equivalence of systems) If $\Gamma; \Delta \Rightarrow \delta$ is derivable in **ILP** then $\Gamma, \Delta \Rightarrow_{\mathtt{p}} \delta$ is derivable according to the rules of figures 3.1 and 8.1. Conversely, if $\Gamma \Rightarrow_{\mathtt{d}} \gamma$ or $\Delta \Rightarrow_{\mathtt{p}} \delta$ is derivable in the $(\circ, \epsilon, \mathbb{J})$-free fragment of the sequent calculus then $\Gamma; \_ \Rightarrow \gamma$ or $\_; \Delta \Rightarrow \delta$ is derivable in **ILP**, respectively.

Remark 8.2.2 The difficult part is to show that every **ILP** derivation can be transformed into a derivation in the $(\circ, \epsilon, \mathbb{J})$-free fragment of the sequent calculus presented in this work. We pro-

$$\frac{}{\eta_p \Rightarrow_{\mathtt{d}} \eta_p}\,\mathtt{ax} \qquad \frac{\Gamma_1 \Rightarrow_{\mathtt{d}} \gamma' \quad \gamma', \Gamma_2 \Rightarrow_{\mathtt{d}} \gamma}{\Gamma_1, \Gamma_2 \Rightarrow_{\mathtt{d}} \gamma}\,\mathtt{cut}$$

$$\frac{\gamma', \gamma', \Gamma \Rightarrow_{\mathtt{d}} \gamma}{\gamma', \Gamma \Rightarrow_{\mathtt{d}} \gamma}\,\mathtt{con} \qquad \frac{\eta_{p_1}, \ldots, \eta_{p_n}, \Xi \Rightarrow_{\mathtt{d}} \eta_p}{\eta^{\circ}_{p_1}, \ldots, \eta^{\circ}_{p_n}, \Xi \Rightarrow_{\mathtt{d}} \eta^{\circ}_p}\,(-)^{\circ}$$

$$\frac{\Gamma_1 \Rightarrow_{\mathtt{d}} \eta_p \quad \xi, \Gamma_2 \Rightarrow_{\mathtt{d}} \gamma}{\eta_p \supset \xi, \Gamma_1, \Gamma_2 \Rightarrow_{\mathtt{d}} \gamma}\,\mathcal{L}\supset \qquad \frac{\Gamma, \eta_p \Rightarrow_{\mathtt{d}} \xi}{\Gamma \Rightarrow_{\mathtt{d}} \eta_p \supset \xi}\,\mathcal{R}\supset$$

Figure 8.3: An intermediate system

ceed essentially in two stages. First, we introduce an intermediate system consisting of the rules in figures 8.3 and 8.1 and show that every **ILP** derivation can be transformed into a derivation in the intermediate system. Second, we show that every derivation in the intermediate system can be transformed into a derivation obtained by instances of the rules in figures 3.1 and 8.1 only.

Definition 8.2.3 In **ILP**, instances of $\mathtt{ax}$, $\mathtt{cut_d}$, $\mathtt{con_d}$, $\supset/\multimap$, $\mathcal{L}\supset$ and $\mathcal{R}\supset$ are called *causal-deontic*; instances of $\mathbb{U}\,\mathtt{ax}$, $\mathtt{cut_p}$, $\mathtt{con_p}$, $\mathcal{L}\cap$, $\mathcal{R}\cap$, $\mathcal{L}\cup$, $\mathcal{R}\cup$, $\mathcal{L}\supset$ and $\mathcal{R}\supset$ are called *pragmatic*.

Definition 8.2.4 An **ILP** derivation is in *split form* if every branch of the prooftree can be divided into three segments such that, going from the root to the leaf, (1) the first segment is induced by instances of pragmatic rules only, (2) the second segment is induced by instances of the $\mathtt{perm}$ rule only, and (3) the third segment is induced by instances of causal-deontic rules only.

Remark 8.2.5 In order to deal with the first stage of the proof we have introduced the concept of split **ILP** derivation. Its definition generalises the concept of quasi-canonical proof given in [BR03]. Transforming a split **ILP** derivation into a derivation in the intermediate system is straightforward: figure 8.4 gives a schematic representation of the process. Thus, to accomplish the first stage of the proof it remains to show that every **ILP** derivation can be transformed into a split **ILP** derivation. This is a consequence of the two technical lemmas 8.2.7 and 8.2.8 below which depend heavily on the notion of critical sequent.

Definition 8.2.6 In an **ILP** derivation, a sequent S is called *quasi-critical* if one of the following two conditions is satisfied:

$$
\begin{array}{c}
[\text{causal-deontic rules}] \\
\vdots \\
\dfrac{\Gamma, \Gamma'; \_ \Rightarrow \gamma}{\Gamma; \Gamma' \Rightarrow \gamma}\ \texttt{perm} \\
\vdots \\
[\text{pragmatic rules}]
\end{array}
\quad \leadsto \quad
\begin{array}{c}
[\text{causal-deontic rules}] \\
\vdots \\
\dfrac{\dfrac{\Gamma, \Gamma'; \_ \Rightarrow \gamma}{\Gamma; \Gamma' \Rightarrow \gamma}\ \texttt{perm}}{\_; \Gamma, \Gamma' \Rightarrow \gamma}\ \texttt{perm} \\
\vdots \\
[\text{pragmatic rules}]
\end{array}
\quad \leadsto \quad
\begin{array}{c}
[\text{intermediate rules}] \\
\vdots \\
\dfrac{\Gamma, \Gamma' \Rightarrow_{\mathtt{d}} \gamma}{\Gamma, \Gamma' \Rightarrow_{\mathtt{p}} \gamma}\ (\mathtt{d/p}) \\
\vdots \\
[\text{pragmatic rules}]
\end{array}
$$

Figure 8.4: Translation of derivations

1. $\mathcal{S}$ is the conclusion of a `perm` rule and there is at least one instance of a pragmatic rule above `perm` in the prooftree;

2. $\mathcal{S}$ is the conclusion of a causal-deontic rule $\mathcal{R}$ and there is at least one instance of a pragmatic or `perm` rule above $\mathcal{R}$ in the prooftree.

The sequent $\mathcal{S}$ is called *critical* if $\mathcal{S}$ is quasi-critical and, further, the derivations of the premises of either `perm` or $\mathcal{R}$ are in split form.

Lemma 8.2.7 If a critical sequent $\mathcal{S}$ is the conclusion of an instance of $\mathcal{R}\supset$ or $\supset/\multimapinv$ then there exists a split **ILP** derivation of it.

Proof Due to the definition of critical sequent, the premise of such a rule can only be an instance of $\mathtt{cut_p}$ which empties the pragmatic area. So, in order to get a split derivation we can essentially use an argument analogous to cut elimination. Before we give the details of the proof, let us point out the particular form of the premises of $\mathtt{cut_p}$ in this situation: the left one is such that both areas of the antecedent are empty, the right one such that exactly one formula occurs in the pragmatic area. As a consequence we have that all cuts in the inductive step are logical, i.e. the cut formula is principal in both premises. Thus we have to consider only the following two distinct cases.

1. If the right premise of $\mathtt{cut_p}$ is the conclusion of an instance of `perm` then

$$
\dfrac{\begin{array}{c}\pi_1 \\ \_; \_ \Rightarrow \gamma'\end{array} \qquad \dfrac{\begin{array}{c}\pi_2 \\ \Gamma, \gamma'; \_ \Rightarrow \gamma\end{array}}{\Gamma; \gamma' \Rightarrow \gamma}\ \texttt{perm}}{\Gamma; \_ \Rightarrow \gamma}\ \mathtt{cut_p}
$$

reduces to

$$\dfrac{\dfrac{\pi_1}{-;-\Rightarrow\gamma'} \quad \dfrac{\pi_2}{\Gamma,\gamma';-\Rightarrow\gamma}}{\Gamma;-\Rightarrow\gamma}\,\mathtt{cut_d}$$

and, hence, by applying either $\mathcal{R}\supset$ or $\supset/\multimap$, we have a split derivation of the critical sequent we started with.

2. If the left premise of $\mathtt{cut_p}$ is the conclusion of an instance of $\mathcal{R}\supset$ and the right premise of $\mathtt{cut_p}$ is the conclusion of an instance of $\mathcal{L}\supset$ then

$$\dfrac{\dfrac{\dfrac{\pi}{-;\delta_1\Rightarrow\delta_2}}{-;-\Rightarrow\delta_1\supset\delta_2}\,\mathcal{R}\supset \quad \dfrac{\dfrac{\pi_1}{-;-\Rightarrow\delta_1} \quad \dfrac{\pi_2}{\Gamma;\delta_2\Rightarrow\gamma}}{\Gamma;\delta_1\supset\delta_2\Rightarrow\gamma}\,\mathcal{L}\supset}{\Gamma;-\Rightarrow\gamma}\,\mathtt{cut_p}$$

reduces to

$$\dfrac{\dfrac{\dfrac{\pi_1}{-;-\Rightarrow\delta_1} \quad \dfrac{\pi}{-;\delta_1\Rightarrow\delta_2}}{-;-\Rightarrow\delta_2}\,\mathtt{cut_p} \quad \dfrac{\pi_2}{\Gamma;\delta_2\Rightarrow\gamma}}{\Gamma;-\Rightarrow\gamma}\,\mathtt{cut_p}$$

where both cuts have smaller rank. Hence, by applying either $\mathcal{R}\supset$ or $\supset/\multimap$, we have a split derivation of the critical sequent we started with. The cases where the left premise is the conclusion of an instance of $\mathcal{R}\cap$ or $\mathcal{R}\cup$ and the right premise is the conclusion of an instance of $\mathcal{L}\cap$ or $\mathcal{R}\cup$, respectively, are similar and therefore omitted. □

Lemma 8.2.8 If a critical sequent $\mathcal{S}$ is not the conclusion of an instance of $\mathcal{R}\supset$ or $\supset/\multimap$ then there exists a split **ILP** derivation of it.

Proof If the critical sequent is the conclusion of an instance of $\mathtt{cut_d}$ or $\mathcal{L}\supset$ then the proof is by induction on the height of the derivation of the right premise. Note that if the right premise of $\mathtt{cut_d}$ is the conclusion of an instance of `perm` then we can apply lemma 8.2.9 below. If the critical sequent is the conclusion of an instance of `perm` or $\mathtt{con_d}$ then the proof is by induction on the height of the derivation of the premise. For illustrative purposes let us consider the following case: if the right premise of $\mathtt{cut_d}$ is the conclusion of an instance of $\mathcal{L}\supset$ then

$$\dfrac{\dfrac{\pi}{\Gamma_1;\delta_1\supset\delta_2,\Delta\Rightarrow\gamma} \quad \dfrac{\dfrac{\pi_1}{-;\Delta\Rightarrow\delta_1} \quad \dfrac{\pi_2}{\gamma,\Gamma_2;\delta_2,\Delta\Rightarrow\delta}}{\gamma,\Gamma_2;\delta_1\supset\delta_2,\Delta\Rightarrow\delta}\,\mathcal{L}\supset}{\Gamma_1,\Gamma_2;\delta_1\supset\delta_2,\Delta\Rightarrow\delta}\,\mathtt{cut_d}$$

reduces to

$$\dfrac{\begin{matrix}\pi_1' \\ -;\delta_1 \supset \delta_2, \Delta \Rightarrow \delta_1\end{matrix} \qquad \dfrac{\begin{matrix}\pi' \\ \Gamma_1;\delta_2,\delta_1 \supset \delta_2, \Delta \Rightarrow \gamma\end{matrix} \qquad \begin{matrix}\pi_2' \\ \gamma, \Gamma_2;\delta_2,\delta_1 \supset \delta_2, \Delta \Rightarrow \delta\end{matrix}}{\Gamma_1,\Gamma_2;\delta_2,\delta_1 \supset \delta_2, \Delta \Rightarrow \delta}\,\mathtt{cut_d}}{\dfrac{\Gamma_1,\Gamma_2;\delta_1 \supset \delta_2,\delta_1 \supset \delta_2, \Delta \Rightarrow \delta}{\Gamma_1,\Gamma_2;\delta_1 \supset \delta_2, \Delta \Rightarrow \delta}\,\mathtt{con_p}}\,\mathcal{L}\supset$$

where π', π_1', and π_2' are obtained from π, π_1, and π_2 by weakening. Since depth-preserving weakening is admissible in **ILP** we can apply the induction hypothesis.[1] □

Lemma 8.2.9 If a critical sequent $\Gamma_1, \Gamma_2; \Gamma, \Delta \Rightarrow \gamma$ is the conclusion of an instance of $\mathtt{cut_d}$ and the right premise of $\mathtt{cut_d}$ is the conclusion of an instance of $\mathtt{perm}$

$$\dfrac{\begin{matrix}\pi_1 \\ \Gamma_1;\Gamma,\Delta \Rightarrow \gamma'\end{matrix} \qquad \begin{matrix}\pi_2 \\ \gamma',\Gamma_2,\Gamma;\_ \Rightarrow \gamma \\ \hline\hline \gamma',\Gamma_2;\Gamma,\Delta \Rightarrow \gamma\end{matrix}\,\mathtt{perm}}{\Gamma_1,\Gamma_2;\Gamma,\Delta \Rightarrow \gamma}\,\mathtt{cut_d}$$

then there exists a split **ILP** derivation of it.

Proof By induction on the height of the derivation of the left premise. For illustrative purposes let us consider the following case: if the left premise of $\mathtt{cut_d}$ is the conclusion of an instance of $\mathcal{L}\supset$ then

$$\dfrac{\dfrac{\begin{matrix}\pi_{11} \\ \_;\Gamma,\Delta' \Rightarrow \delta_1\end{matrix} \qquad \begin{matrix}\pi_{12} \\ \Gamma_1;\delta_2,\Gamma,\Delta' \Rightarrow \gamma'\end{matrix}}{\Gamma_1;\Gamma,\delta_1 \supset \delta_2,\Delta' \Rightarrow \gamma'}\,\mathcal{L}\supset \qquad \begin{matrix}\pi_2 \\ \gamma',\Gamma_2,\Gamma;\_ \Rightarrow \gamma \\ \hline\hline \gamma',\Gamma_2;\Gamma,\delta_1 \supset \delta_2,\Delta' \Rightarrow \gamma\end{matrix}\,\mathtt{perm}}{\Gamma_1,\Gamma_2;\Gamma,\delta_1 \supset \delta_2,\Delta' \Rightarrow \gamma}\,\mathtt{cut_d}$$

reduces to

$$\dfrac{\begin{matrix}\pi_{11} \\ \_;\Gamma,\Delta' \Rightarrow \delta_1\end{matrix} \qquad \dfrac{\begin{matrix}\pi_{12} \\ \Gamma_1;\Gamma,\delta_2,\Delta' \Rightarrow \gamma'\end{matrix} \qquad \begin{matrix}\pi_2 \\ \gamma',\Gamma_2,\Gamma;\_ \Rightarrow \gamma \\ \hline\hline \gamma',\Gamma_2;\Gamma,\delta_2,\Delta' \Rightarrow \gamma\end{matrix}\,\mathtt{perm}}{\Gamma_1,\Gamma_2;\Gamma,\delta_2,\Delta' \Rightarrow \gamma}\,\mathtt{cut_d}}{\Gamma_1,\Gamma_2;\Gamma,\delta_1 \supset \delta_2,\Delta' \Rightarrow \gamma}\,\mathcal{L}\supset$$

and the induction hypothesis applies. □

[1] Depth-preserving weakening is a consequence of the way we have defined the perm rule: we refer to [TS00] for a formal argument. Its intuitive meaning is that we can apply weakening without changing the depth of the prooftree.

Remark 8.2.10 We can now turn to the second stage of the proof of theorem 8.2.1. This is basically accomplished by the following lemma.

Lemma 8.2.11 If $\Delta \Rightarrow_x \delta$ is derivable in the intermediate system then it is derivable according to the rules of figures 3.1 and 8.1.

Proof We consider instances of intermediate rules $\mathcal{R}$ that satisfy the following two conditions:

1. the principal formula of $\mathcal{R}$ does not contain subformulae of the form η_p°;
2. the formula in the succedent of the conclusion of $\mathcal{R}$ is of the form η_p°.

If we take the topmost of these instances and push it upwards in the prooftree then, by iterated application of this process, we can bring all instances of purely causal rules such as $\mathcal{L}\supset$ above the instances of $(-)^\circ$ in the prooftree. For illustrative purposes let us consider the following two cases involving an instance of $\mathcal{L}\supset$ satisfying the above conditions. Since the formula in the succedent of the left premise of $\mathcal{L}\supset$ is not of the form η_p°, we can assume without loss of generality that all formulae in the antecedent are purely causal.

1. $(-)^\circ$ vs. $\mathcal{L}\supset$

$$\dfrac{\begin{matrix}\pi' \\ \Xi' \Rightarrow_d \eta_{p'}\end{matrix} \qquad \dfrac{\begin{matrix}\pi \\ \eta_{p_1}, \ldots, \eta_{p_n}, \xi, \Xi \Rightarrow_d \eta_p\end{matrix}}{\eta_{p_1}^\circ, \ldots, \eta_{p_n}^\circ, \xi, \Xi \Rightarrow_d \eta_p^\circ}(-)^\circ}{\eta_{p_1}^\circ, \ldots, \eta_{p_n}^\circ, \eta_{p'} \supset \xi, \Xi', \Xi \Rightarrow_d \eta_p^\circ}\mathcal{L}\supset$$

permutes to

$$\dfrac{\dfrac{\begin{matrix}\pi' \\ \Xi' \Rightarrow_d \eta_{p'}\end{matrix} \qquad \begin{matrix}\pi \\ \eta_{p_1}, \ldots, \eta_{p_n}, \xi, \Xi \Rightarrow_d \eta_p\end{matrix}}{\eta_{p_1}, \ldots, \eta_{p_n}, \eta_{p'} \supset \xi, \Xi', \Xi \Rightarrow_d \eta_p}\mathcal{L}\supset}{\eta_{p_1}^\circ, \ldots, \eta_{p_n}^\circ, \eta_{p'} \supset \xi, \Xi', \Xi \Rightarrow_d \eta_p^\circ}(-)^\circ$$

2. cut vs. $\mathcal{L}\supset$

$$\dfrac{\begin{matrix}\pi \\ \Xi \Rightarrow_d \eta_{p'}\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1 \\ \xi, \Gamma_1 \Rightarrow_d \gamma\end{matrix} \qquad \begin{matrix}\pi_2 \\ \gamma, \Gamma_2 \Rightarrow_d \eta_p^\circ\end{matrix}}{\xi, \Gamma_1, \Gamma_2 \Rightarrow_d \eta_p^\circ}\text{cut}}{\eta_{p'} \supset \xi, \Xi, \Gamma_1, \Gamma_2 \Rightarrow_d \eta_p^\circ}\mathcal{L}\supset$$

permutes to

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \Xi \Rightarrow_d \eta_{p'}\end{matrix} \qquad \begin{matrix}\pi_1 \\ \xi, \Gamma_1 \Rightarrow_d \gamma\end{matrix}}{\eta_{p'} \supset \xi, \Xi, \Gamma_1 \Rightarrow_d \gamma}\mathcal{L}\supset \qquad \begin{matrix}\pi_2 \\ \gamma, \Gamma_2 \Rightarrow_d \eta_p^\circ\end{matrix}}{\eta_{p'} \supset \xi, \Xi, \Gamma_1, \Gamma_2 \Rightarrow_d \eta_p^\circ}\text{cut}$$

It is then easy to see that we can obtain the desired derivation of the sequent we started with by inserting instances of (a/c) and (c/d) in the appropriate places. □

8.3 Discussion and final remarks

We have considered the full intuitionistic fragment of the logic for pragmatics and shown how the system presented in this thesis relates to the system originally proposed by Bellin and Dalla Pozza in [BD02]. Note that there are two issues that seem worth to be investigated. First, although it is fairly easy to provide a categorical semantics for the full intuitionistic fragment, we have discovered that, rather surprisingly, there are problems with the definition of the Kripke semantics. We have outlined a possible solution to the problem at the end of remark 8.1.4 but it seems that one should look for something better. Second, it would be interesting to see what categorical structures are needed to interpret the original system of Bellin and Dalla Pozza and to compare it with the semantics proposed in this thesis.

Chapter 9

Conclusion

In this thesis we have developed a sound and complete categorical semantics for the full intuitionistic fragment of the logic for pragmatics with assertions, obligations, and causal implication originally proposed by Bellin and Dalla Pozza in [BD02] and an extension thereof with conditional obligations or hypothetical norms. Along the way we have also investigated its proof and type theory, provided a class of sound and complete Kripke models, and shown that the Kripke models can be regarded as a degenerate instance of the categorical semantics by relating them to a class of sound and complete algebraic models. Since we have already hinted at omissions and unresolved issues in the final section of almost each chapter let us conclude the thesis with a short list of possible directions for future research.

Deontic logic and paradoxes Although we made an effort to make the philosophical assumptions underlying this body of work as accessible as possible we have not treated the philosophical consequences inherent to it in great detail. In particular, the extension of the framework with the concept of conditional obligation seems to make it possible to tackle certain puzzles (see for instance [McN06, section 4.5]) in deontic logic.

Dealing with the full system Note that, at the end of chapter 2, we have made the simplifying assumption that the radical part α of an elementary formula consists only of atomic propositions p. Giving up that assumption means that one has to take into consideration the full system. One of the key features of the full logic for pragmatics is that the interaction between sentential formulae and propositions is expressed via modal translations from intuitionistic to classical modal logic (see the translation schemes provided in chapter 2). The challenge is thus to investigate a categorical semantics for classical modal logic and for the Gödel-McKinsey-Tarski interpretation of intuitionistic logic in the classical modal logic **S4**.

Towards a theory of actions It would be interesting to investigate the possibility of developing a theory of actions. White's recent work [Whi08] on a categorical formulation of Reiter's solution to the frame problem is a promising step towards that direction. Technically speaking, it is based on fibrations over labelled transition systems and would allow for a more dynamic approach to the logic for pragmatics. Further, White's framework allows one to deal with a wide range of modal operators and thus it seems also to be a good candidate for studying the full logic for pragmatics with the aforementioned reflection principle.

Decidability and complexity issues We have only mentioned in passing the close connection between causal implications or conditional obligations and the Horn fragment of multiplicative linear logic extended with the structural rule of contraction. It is well known that relevant logics behave rather badly with respect to decidability and complexity (see for instance [Res00]), but one might hope that the restrictions imposed by our framework have some positive impact. The most promising line of attack would be to follow [Kan95] and to investigate whether Horn linear logic with contraction can be related to a suitable class of Petri nets, thus providing means to take advantage of the many results available in that field.

Illocutionary logic and perlocutions In [SV85] one can find a whole body of work on illocutionary logic. It should be very useful to compare the set-theoretic approach taken there with the semantic approach taken in this thesis. Furthermore, in the literature on speech act theory (see for instance [Lev83]) there is also the concept of perlocutionary act which denotes the act of producing a certain effect in the hearer by uttering a sentence. Since in the full system it is possible to make a distinction between descriptive and expressive use of pragmatic operators we think that the logic for pragmatics forms a good starting point for investigating this concept, thus providing a framework for the study of intended meaning.

Software agent communication New paradigms in software engineering such as software agents make heavy use of speech act theory in order to investigate possible frameworks for communication between autonomous software agents (see for instance [Lab01] and [MP03]). We think that the logic for pragmatics studied in this thesis could be very useful to provide specification languages for dialogue protocols. One of the limitations of the logic is that it does not deal with many types of pragmatic forces. Thus it would be important to find meaningful patterns of common-sense reasoning and to extend the rudimentary language with new pragmatic operators which allow one to implement these patterns.

Bibliography

[Ans00] G. E. M. Anscombe. *Intention.* Harvard University Press, second edition, 2000.

[Aus76] J. L. Austin. *How to do Things with Words.* Oxford University Press, second edition, 1976.

[BA08] C. Biasi and F. Aschieri. A term assignment for polarized bi-intuitionistic logic and its strong normalization. *Fundamenta Informaticae*, 84(2):185–205, 2008.

[BB04] G. Bellin and C. Biasi. Towards a logic for pragmatics: assertions and conjectures. *Journal of Logic and Computation*, 14(4):473–506, 2004.

[BCS02] R. Blute, J. R. B. Cockett, and R. A. G. Seely. The logic of linear functors. *Mathematical Structures in Computer Science*, 12(4):513–539, 2002.

[BD02] G. Bellin and C. Dalla Pozza. A pragmatic interpretation of substructural logics. In W. Sieg and C. Talcott, editors, *Reflections on the Foundations of Mathematics, Essays in Honor of Solomon Feferman*, volume 15 of *Lecture Notes in Logic*, pages 139–163. ASL, 2002.

[BdPR01] G. Bellin, V. C. V. de Paiva, and E. Ritter. Extended Curry-Howard correspondence for a basic constructive modal logic. Preprint available online at `http://profs.sci.univr.it/~bellin/papers.html`, 2001.

[Bel85] G. Bellin. A system of natural deduction for GL. *Theoria*, 2:89–114, 1985.

[Ben94] P. N. Benton. A mixed linear and non-linear logic: proofs, terms and models. In L. Pacholski and J. Tiuryn, editors, *Computer Science Logic 1994*, volume 933 of *Lecture Notes in Computer Science*, pages 121–135. Springer, 1994.

[Bie94] G. M. Bierman. *On Intuitionistic Linear Logic.* PhD thesis, University of Cambridge, 1994.

[BR03] G. Bellin and K. Ranalter. A Kripke-style semantics for the intuitionistic logic of pragmatics ILP. *Journal of Logic and Computation*, 13(5):755–775, 2003.

[Bra96] E. Braun, editor. *Der Paradigmenwechsel in der Sprachphilosophie: Studien und Texte*. Wissenschaftliche Buchgesellschaft, 1996.

[Che80] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.

[Dal91] C. Dalla Pozza. Una interpretazione pragmatica della logica proposizionale intuizionistica. In G. Usberti, editor, *Problemi Fondazionali nella Teoria del Significato*, pages 49–71. Olschki, Florence, 1991.

[Dal97] C. Dalla Pozza. Una logica pragmatica per la concezione "espressiva" delle norme. In A. Martino, editor, *Logica delle Norme*. SEU, Pisa, 1997.

[DG95] C. Dalla Pozza and C. Garola. A pragmatic interpretation of intuitionistic propositional logic. *Erkenntnis*, 43(1):81–109, 1995.

[Epp03] A. Eppendahl. *Categories for Types and Axiomatic Domain Theory*. PhD thesis, University of London, 2003.

[Fre19] G. Frege. Der Gedanke: Eine logische Untersuchung. *Beiträge zur Philosophie des deutschen Idealismus*, 1:58–77, 1918/19. Reprinted as excerpt in [Bra96, pp. 210-218]. See also [Fre56] for an English translation.

[Fre56] G. Frege. The thought: A logical inquiry. *Mind*, 65(259):298–311, 1956.

[Gen35] G. Gentzen. Untersuchungen über das logische Schliessen I, II. *Mathematische Zeitschrift*, 39:176–210 & 405–431, 1935.

[Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[Gir95] J.-Y. Girard. Linear logic: Its syntax and semantics. In J.-Y. Girad, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, pages 1–42. Cambridge University Press, 1995.

[GLT90] J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Cambridge University Press, 1990.

[HHS05] R. Houston, D. Hughes, and A. Schalk. Modelling linear logic without units. Preprint available online at `http://arxiv.org/abs/math.CT/0504037`, 2005.

[Jac93] B. Jacobs. Semantics of lambda-I and of other "substructure" lambda calculi. In M. Bezem and J. F. Groote, editors, *Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 195–208. Springer, 1993.

[Jac94] B. Jacobs. Semantics of weakening and contraction. *Annals of Pure and Applied Logic*, 69(1):73–106, 1994.

[Jac99] B. Jacobs. *Categorical Logic and Type Theory*. North Holland, 1999.

[Kak07] Y. Kakutani. Calculi for intuitionistic normal modal logic. In *Proceedings of Programming and Programming Languages*, 2007.

[Kan94] M. I. Kanovich. The complexity of Horn fragments of linear logic. *Annals of Pure and Applied Logic*, 69(2-3):194–241, 1994.

[Kan95] M. I. Kanovich. Petri nets, Horn programs, linear logic and vector games. *Annals of Pure and Applied Logic*, 75(1-2):107–135, 1995.

[Lab01] Y. Labrou. Standardizing agent communication. In M. Luck, O. Stepankova, V. Marik, and R. Trappl, editors, *Multi-Agent Systems and Applications*, volume 2086 of *Lecture Notes in Computer Science*, pages 74–97. Springer, 2001.

[Lev83] S. C. Levinson. *Pragmatics*. Cambridge University Press, 1983.

[LS86] J. Lambek and P. J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge University Press, 1986.

[Mac98] S. Mac Lane. *Categories for the Working Mathematician*. Springer, second edition, 1998.

[Mai08] M. E. Maietti. Modular correspondence between dependent type theories and categories including pretopoi and topoi. Submitted for publication, 2008.

[MB03] M.-A. Moens and F. Borceux. On Azumaya graphs. *Algebras and Representation Theory*, 6(1):71–95, 2003.

[MBCB02] M.-A. Moens, U. Berni-Canani, and F. Borceux. On regular presheaves and regular semi-categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 43(3):163–190, 2002.

[McN06] P. McNamara. Deontic logic. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. `http://plato.stanford.edu/entries/logic-deontic/`, Spring 2006.

[MdPR00] M. E. Maietti, V. C. V. de Paiva, and E. Ritter. Categorical models for intuitionistic and linear type theory. In J. Tiuryn, editor, *Foundations of Software Science and Computation Structures*, volume 1784 of *Lecture Notes in Computer Science*, pages 223–237, 2000.

[MMdPR05] M. E. Maietti, P. Maneggia, V. C. V. de Paiva, and E. Ritter. Relating categorical semantics for intuitionistic linear logic. *Applied Categorical Structures*, 13(1):1–36, 2005.

[MP03] P. McBurney and S. Parsons. Dialogue game protocols. In M.-P. Huget, editor, *Communication in Multiagent Systems: Agent Communication Languages and Conversation Policies*, volume 2650 of *Lecture Notes in Computer Science*, pages 269–283. Springer, 2003.

[Pea00] J. Pearl. *Causality: Models, Reasoning, and Inference*. Cambridge University Press, 2000.

[POY04] D. J. Pym, P. W. O'Hearn, and H. Yang. Possible worlds and resources: the semantics of BI. *Theoretical Computer Science*, 315(1):257–305, 2004.

[Ran08a] K. Ranalter. A semantic analysis of a logic for pragmatics with assertions, obligations and causal implication. *Fundamenta Informaticae*, 84(3-4):443–470, 2008.

[Ran08b] K. Ranalter. Two-sequent K and simple fibrations (preliminary report). In V. C. V. de Paiva and A. Nanevski, editors, *Proceedings of the Workshop on Intuitionistic Modal Logic and Applications (IMLA '08)*, Microsoft Technical Report MSR-TR-2008-90, 2008.

[Res00] G. Restall. *An Introduction to Substructural Logics*. Routledge, 2000.

[SH00] L. Schröder and H. Herrlich. Free adjunctions of morphisms. *Applied Categorical Structures*, 8(4):595–606, 2000.

[SL95] R. Stuhlmann-Laeisz. *Gottlob Freges 'Logische Untersuchungen': Darstellung und Interpretation.* Wissenschaftliche Buchgesellschaft, 1995.

[SV85] J. R. Searle and D. Vanderveken. *Foundations of Illocutionary Logic.* Cambridge University Press, 1985.

[TS00] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, second edition, 2000.

[Whi08] G. G. White. Davidson and Reiter on actions. *Fundamenta Informaticae*, 84(2):259–289, 2008.